Self-Adapting Networks

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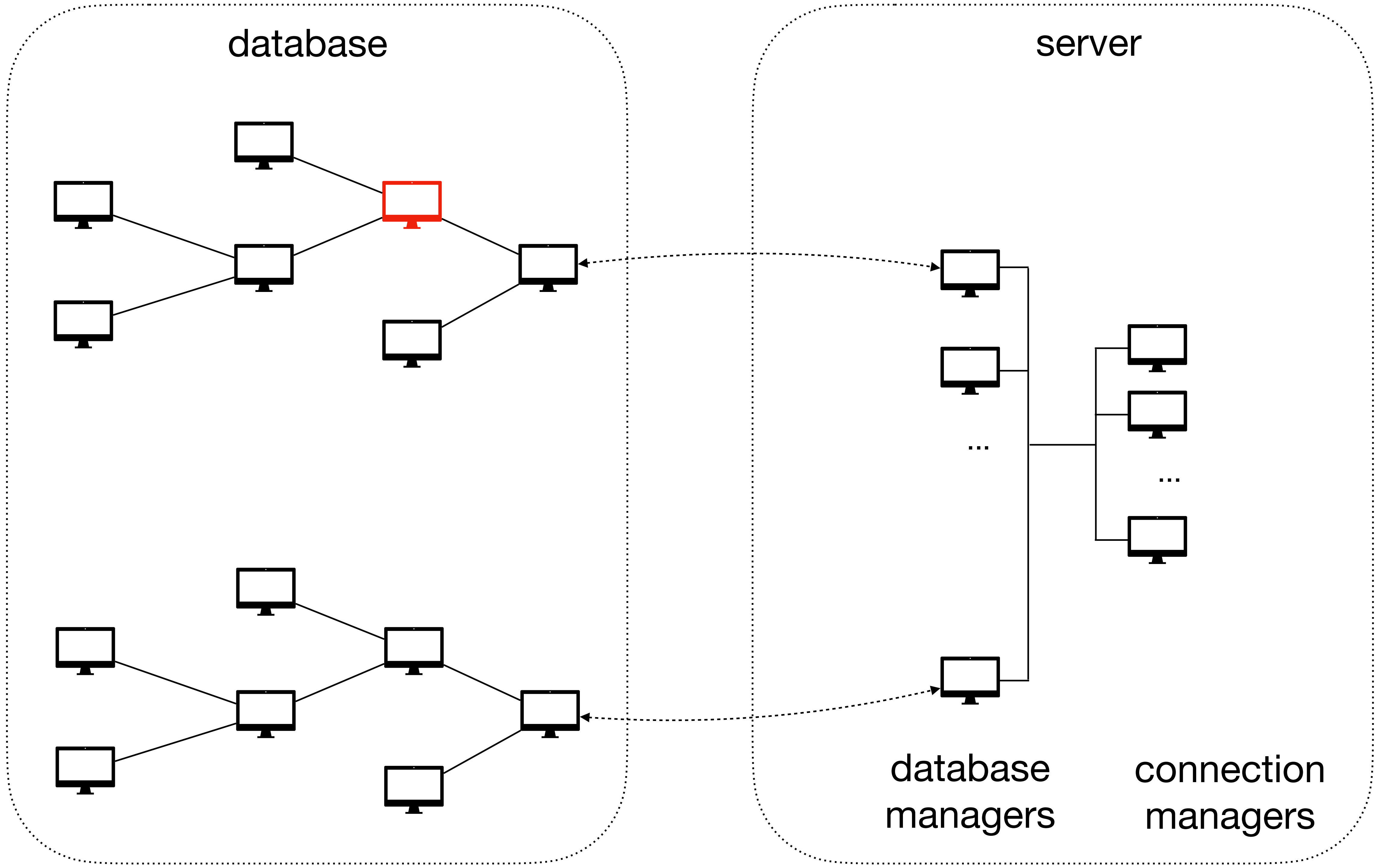
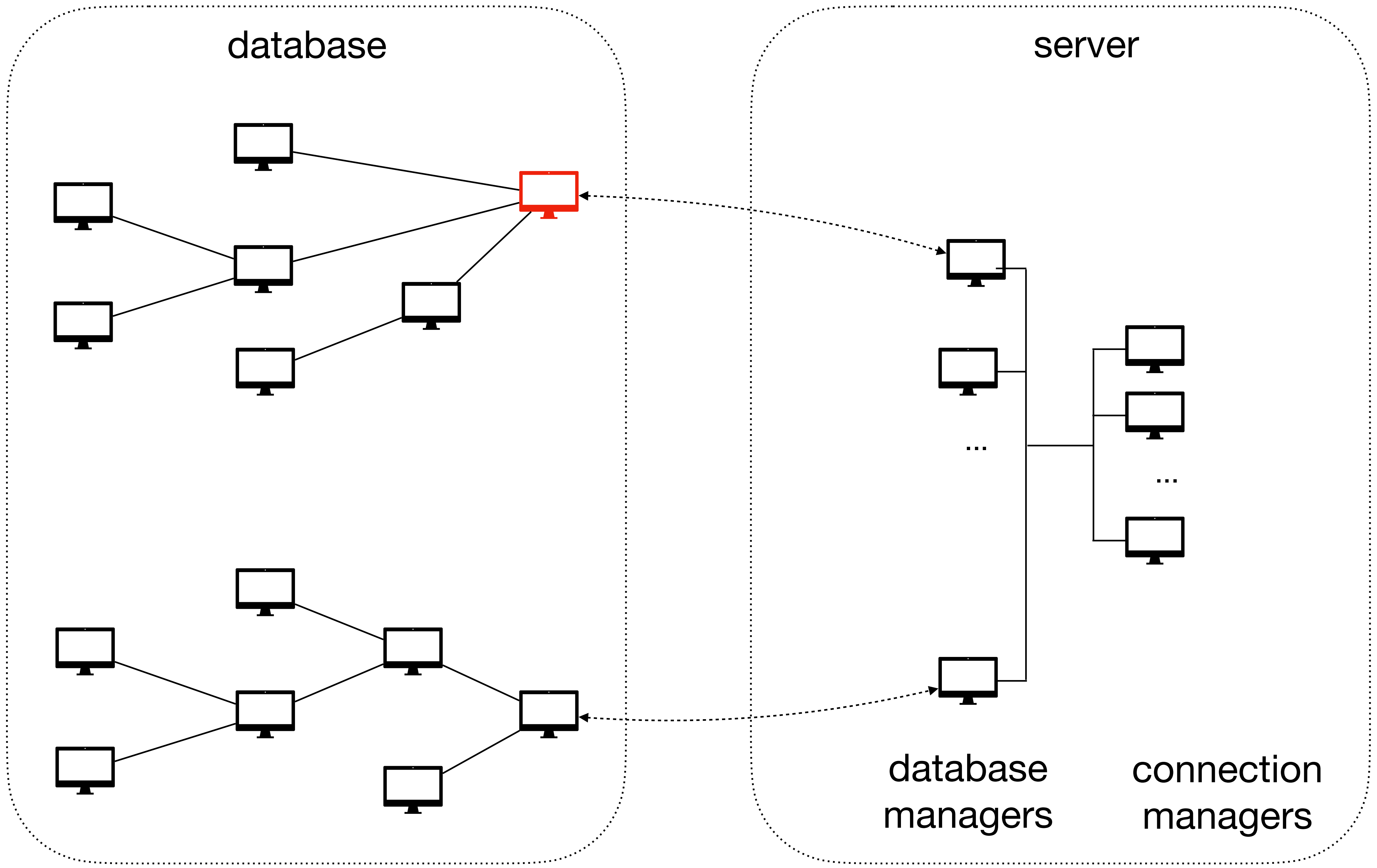
Architectures and Reconfiguration



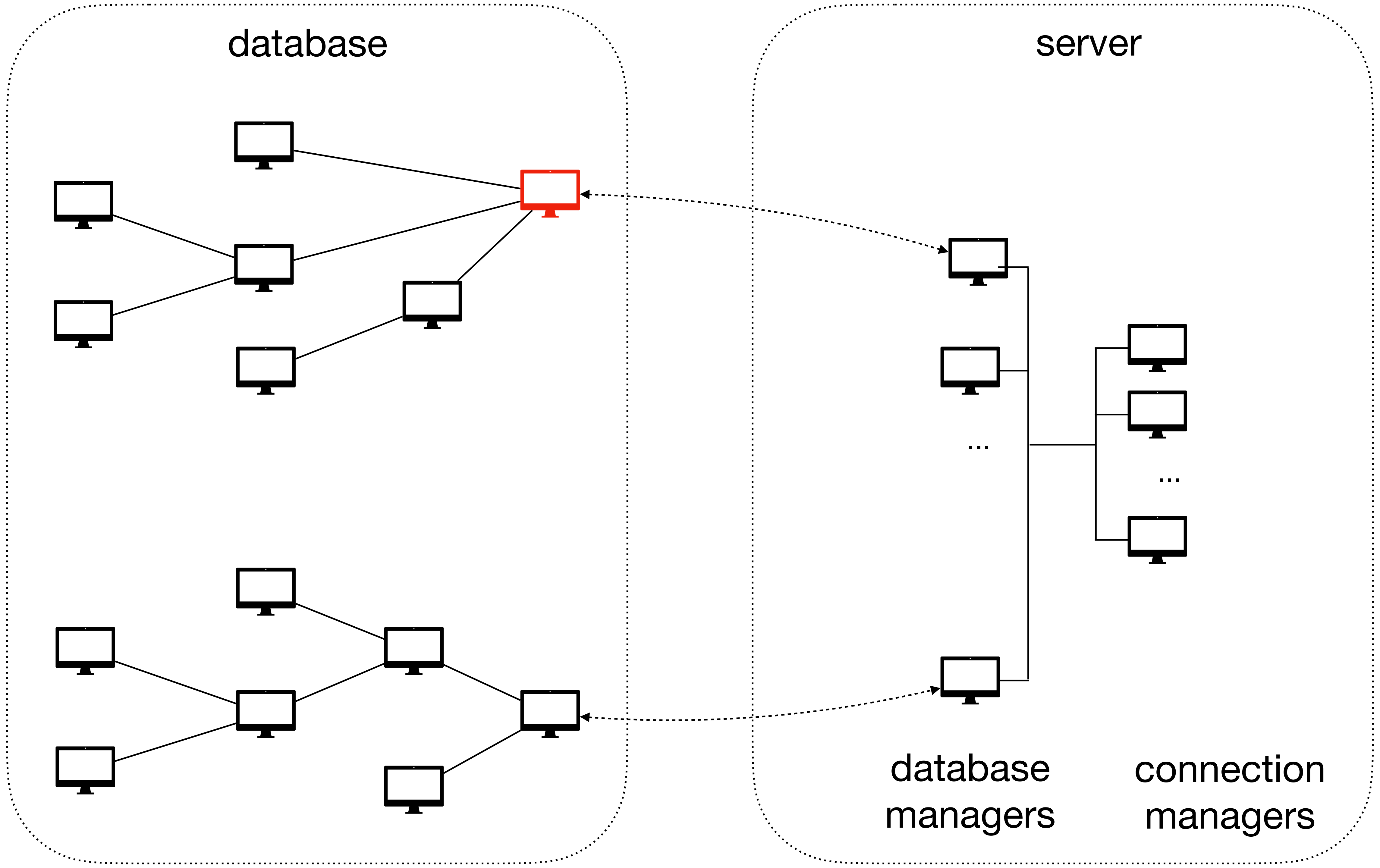
Architectural styles
(pipeline, tree, star, clique, etc.)

Architectures and Reconfiguration



Internal reconfiguration
(self-adapting networks)

Architectures and Reconfiguration



Internal reconfiguration
(self-adapting networks)

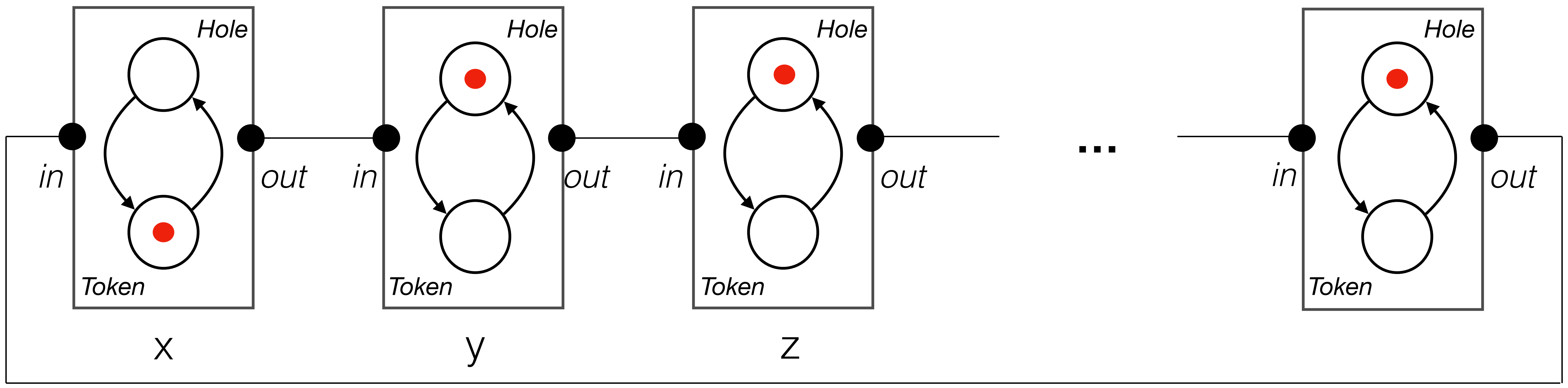
Internal vs external initiation of architectural changes

- self-adapting systems have internal initiation (guards)

Centralized vs distributed management

- centralized (sequential) management: simpler to implement and supported by the majority of dynamic reconfiguration languages
- distributed (parallel) management: efficient and realistic but more challenging to model and reason about

What can possibly go wrong?



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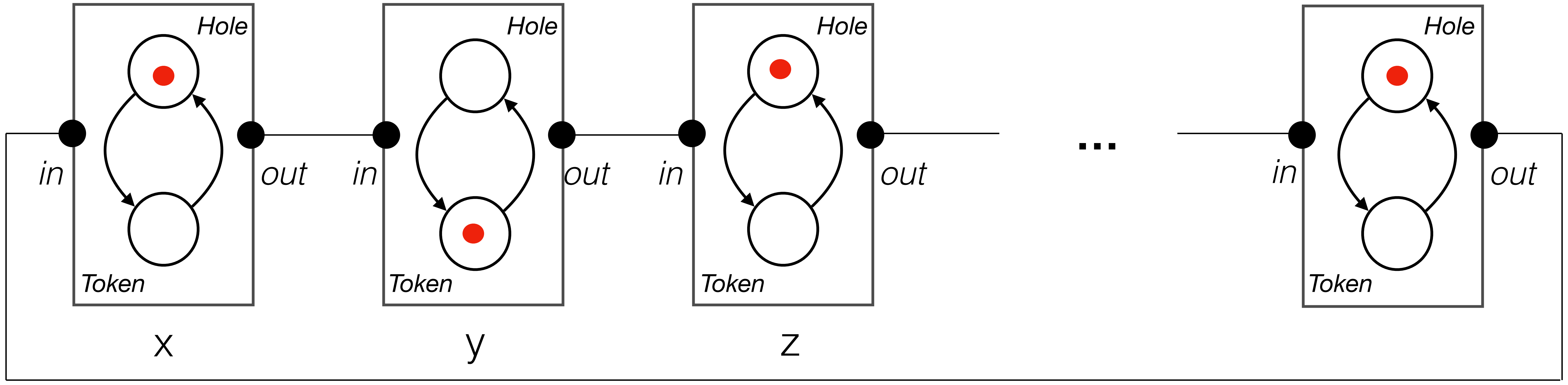

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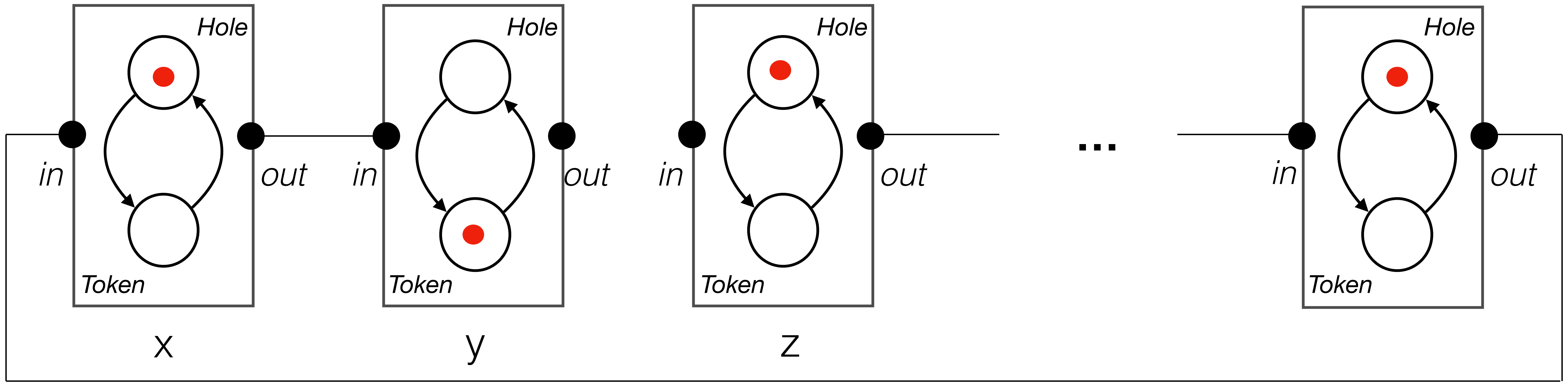

reconfiguration program {

```
disconnect(y.out, z.in);
```

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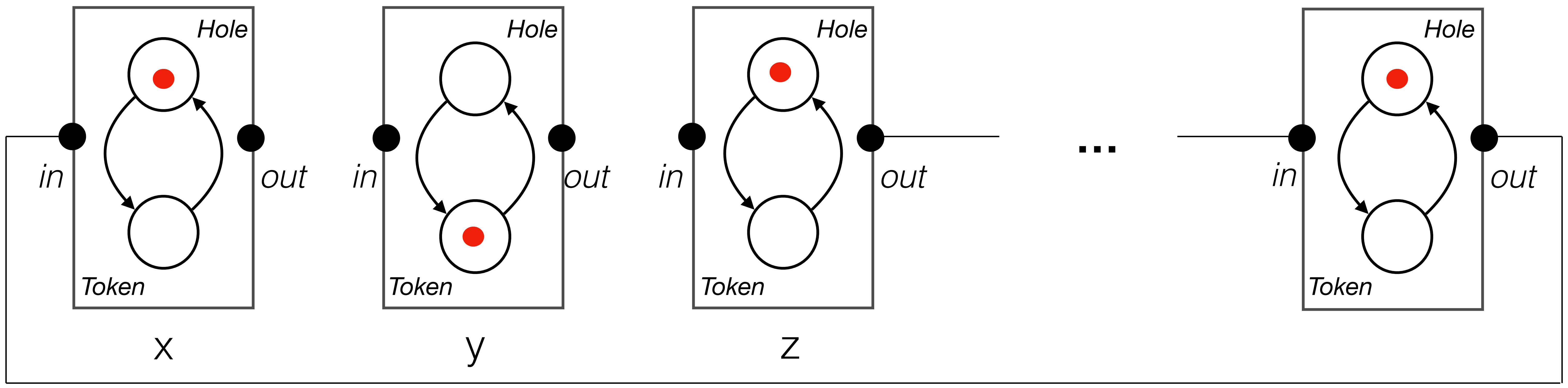

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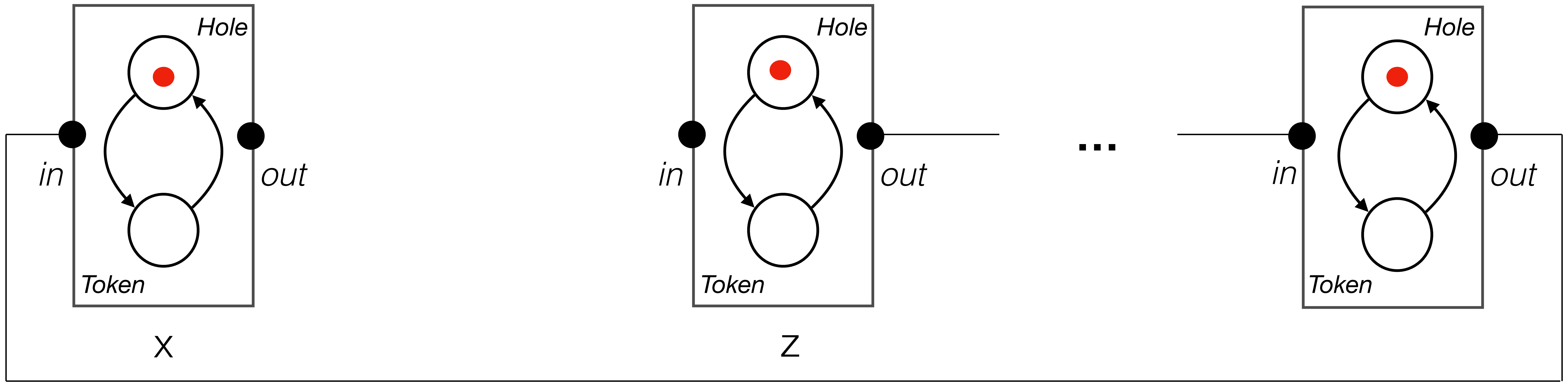

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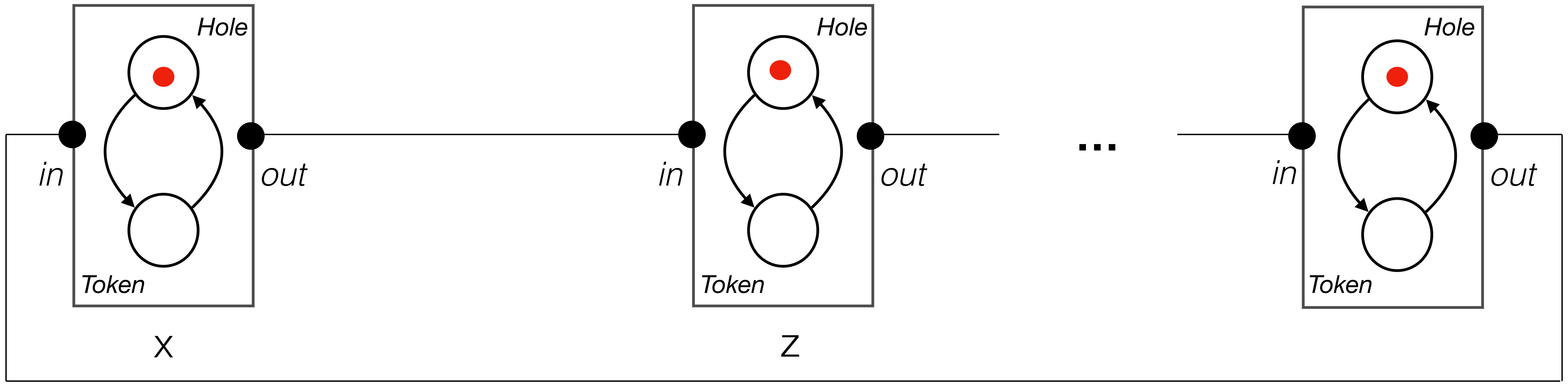

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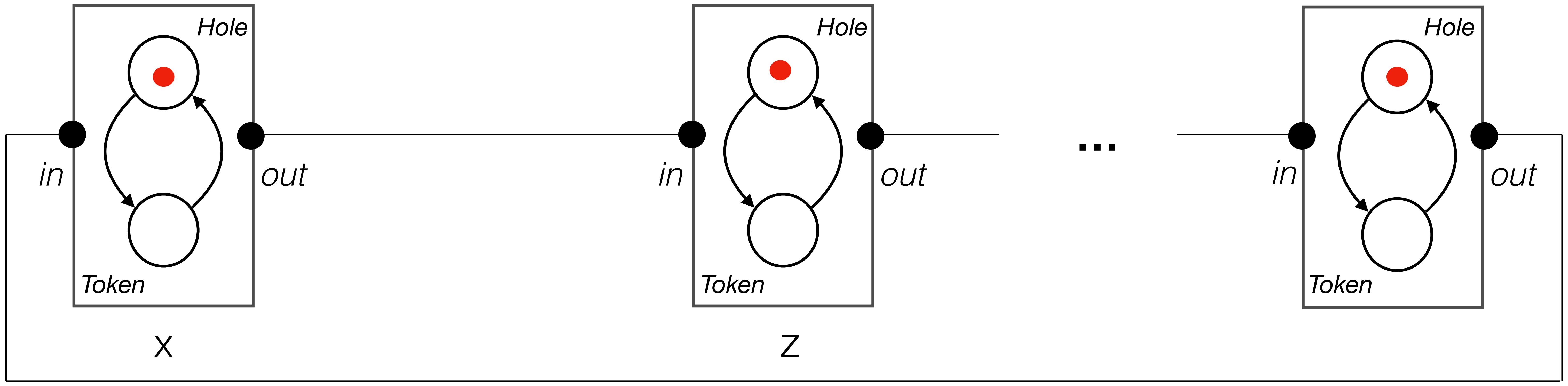

reconfiguration program

```
disconnect(y.out, z.in);
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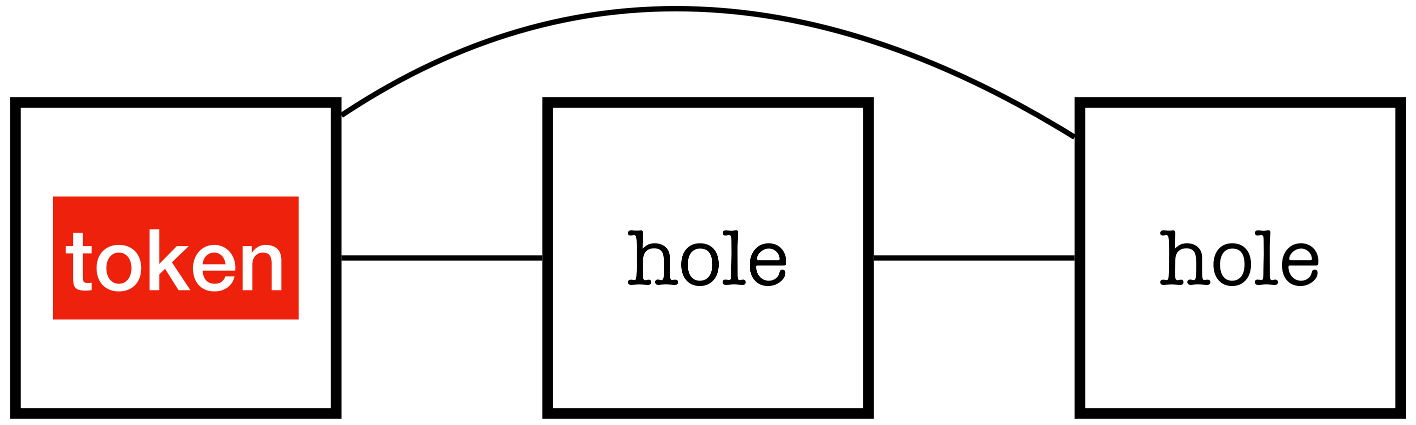
connect(x.out, z.in);
```

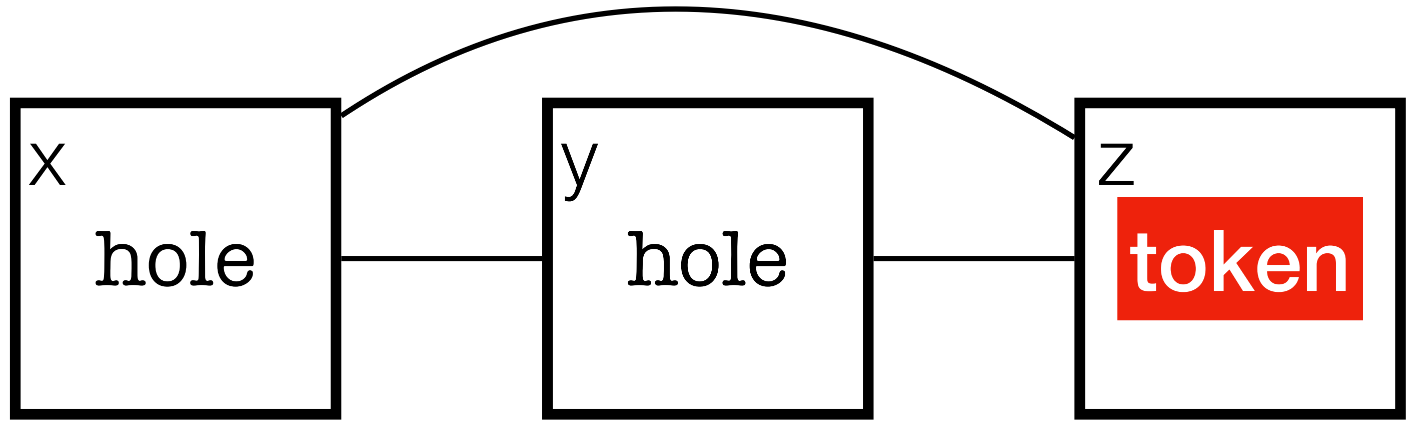
→ deadlock

Network Configurations



A configuration is a network with a snapshot of the states of each component

Havoc vs Reconfiguration Actions



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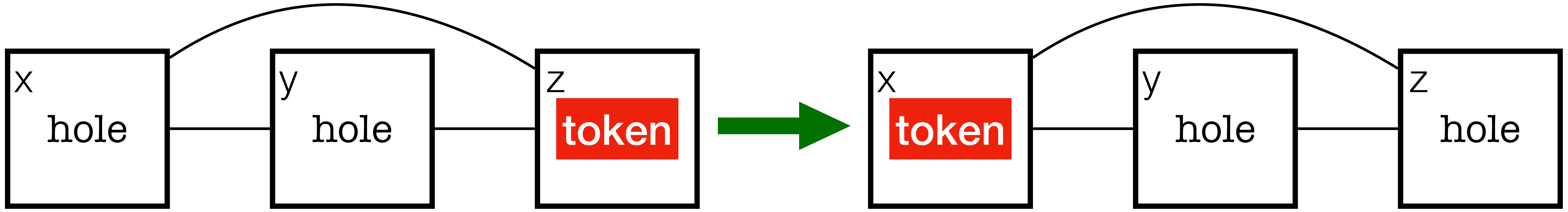

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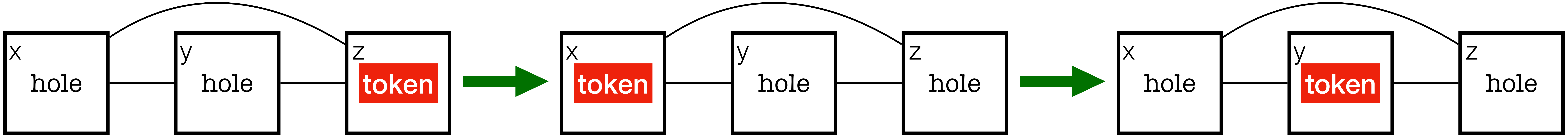

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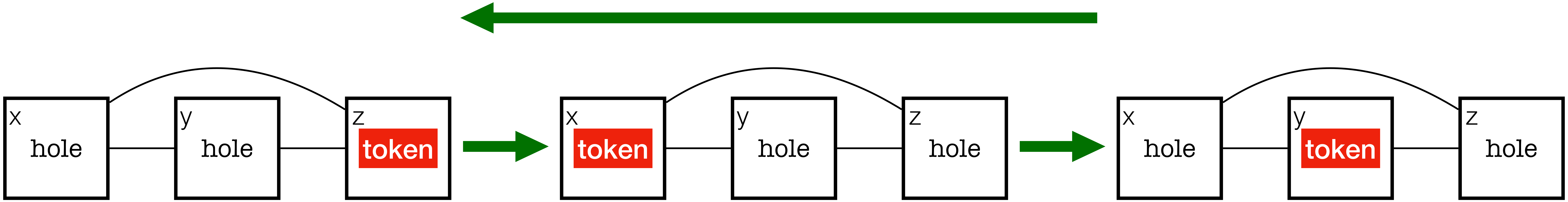

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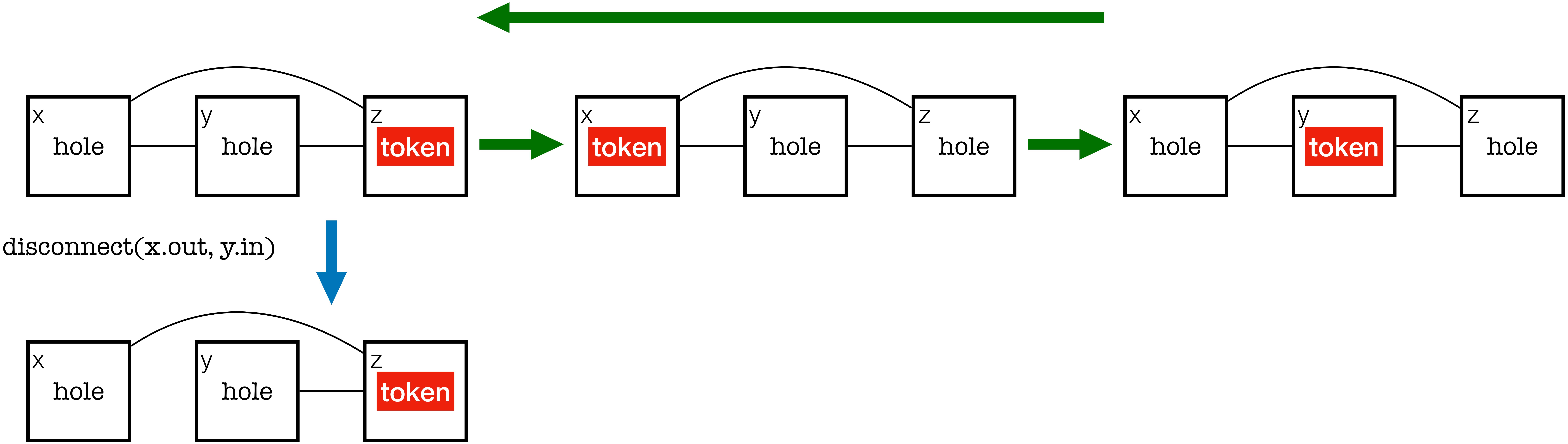

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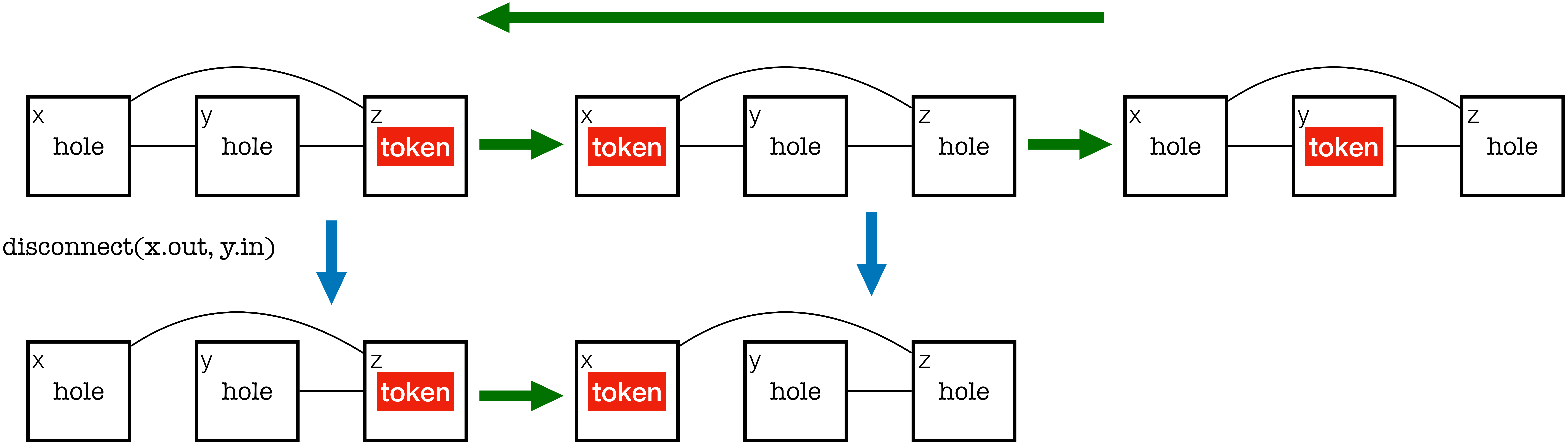

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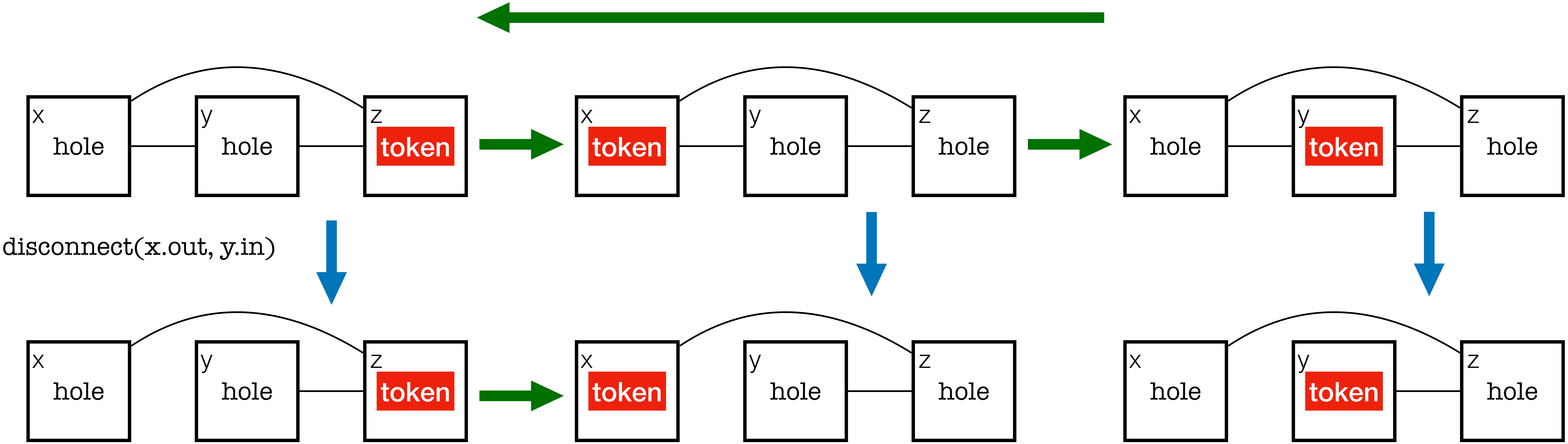

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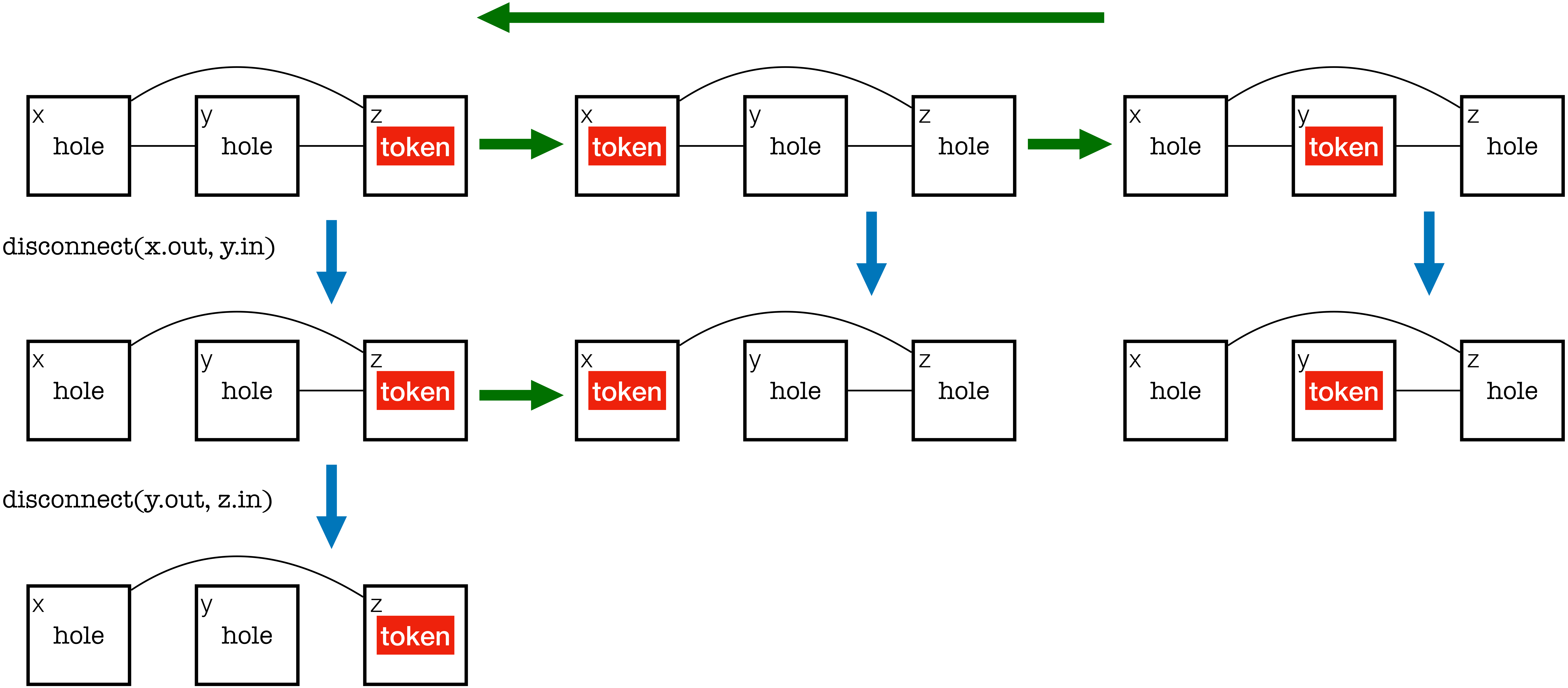

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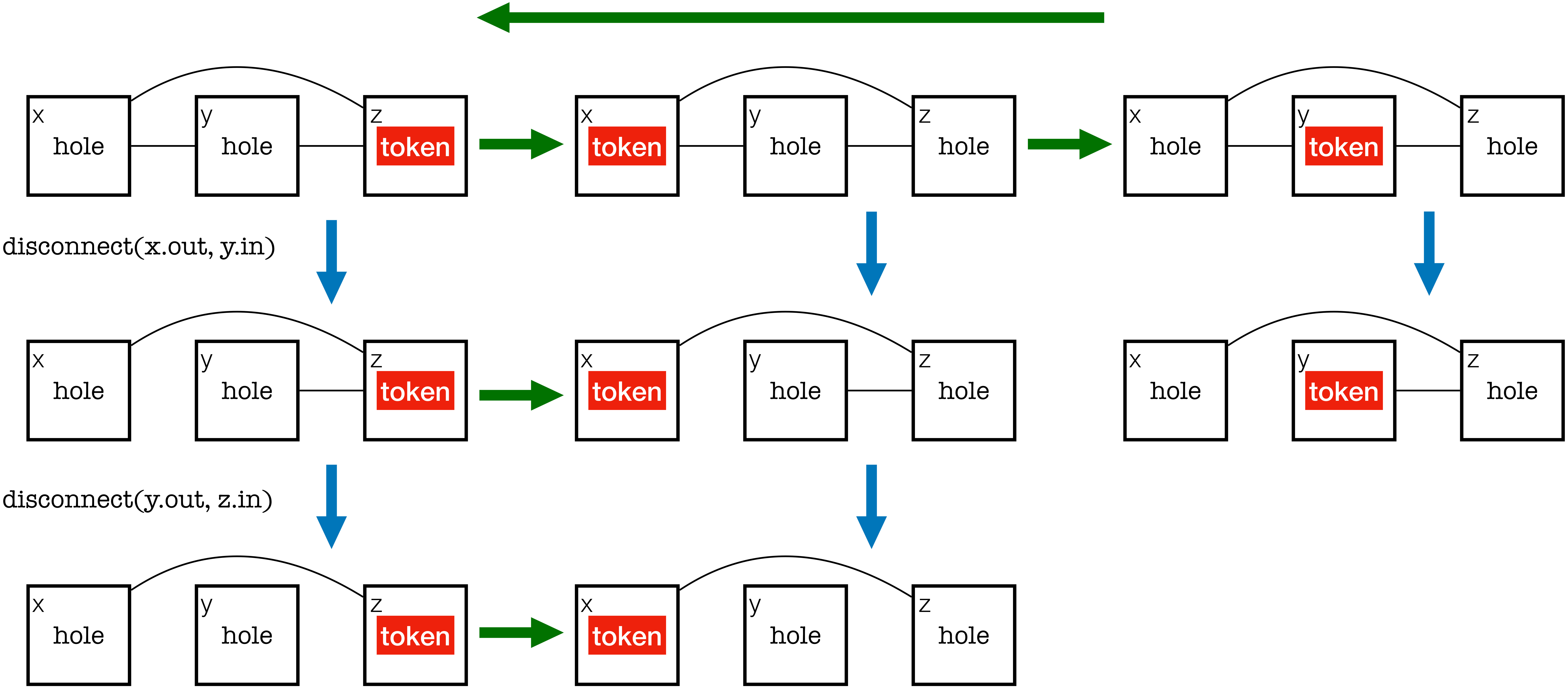

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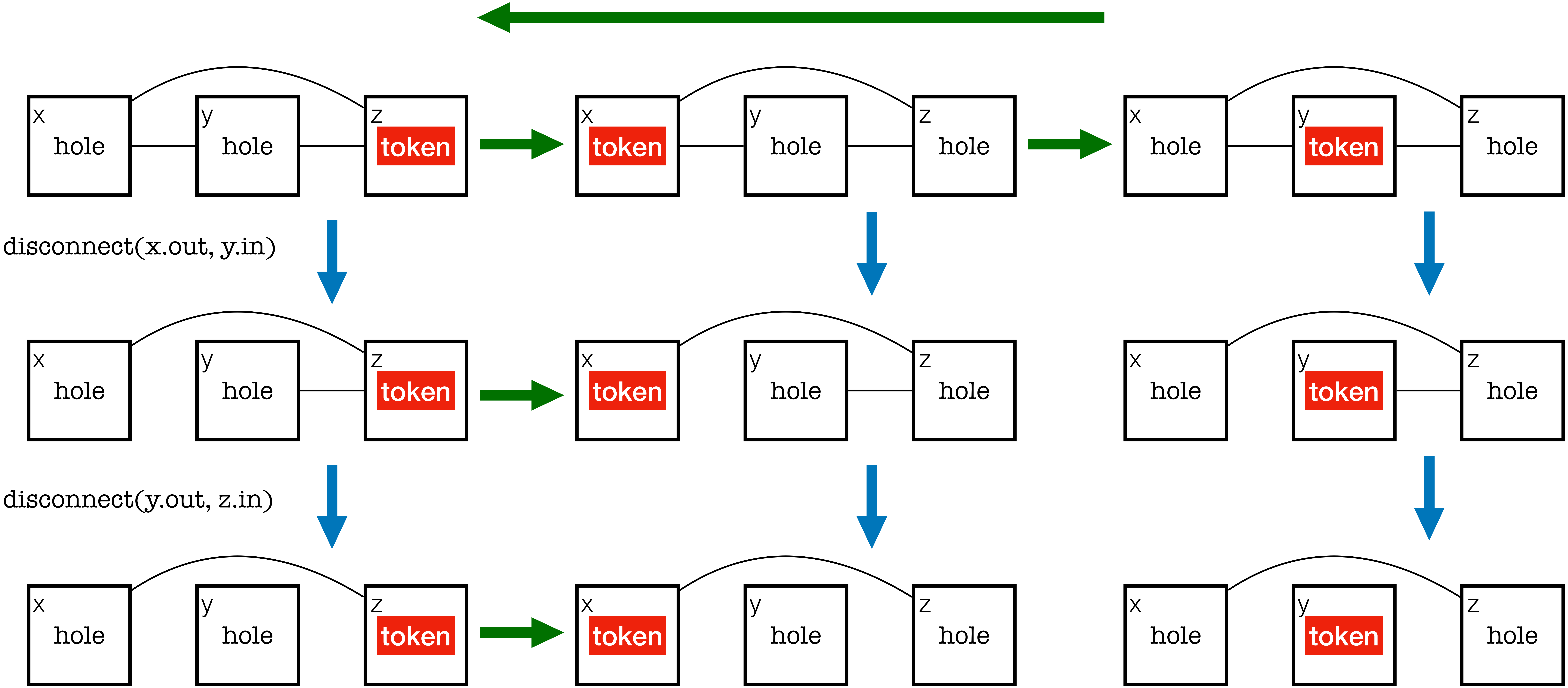

Self-Adapting Networks are Infinite-state Systems

- Transition systems with unbounded number of configurations:
  - new components can be added, yielding increasingly complex reachability graphs

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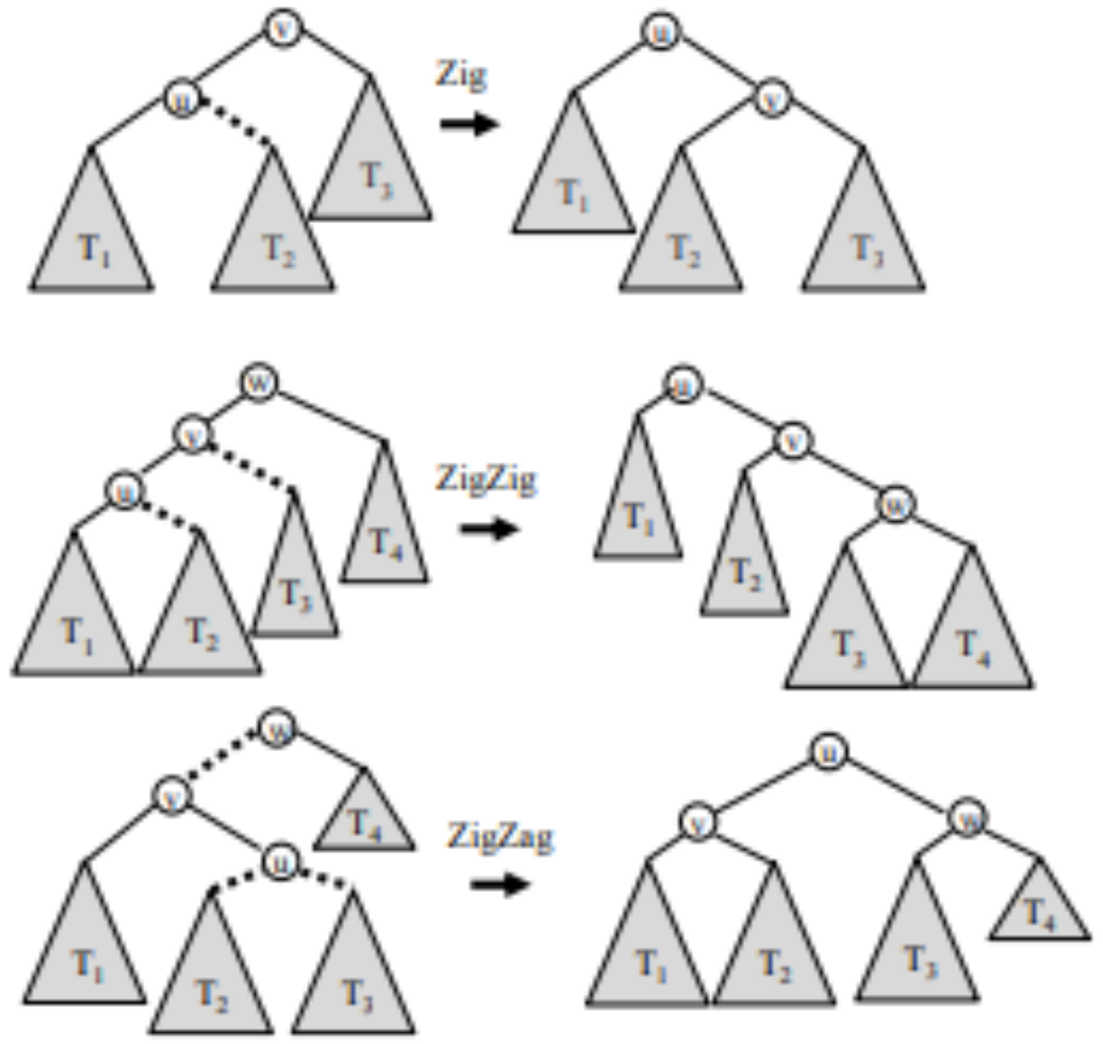
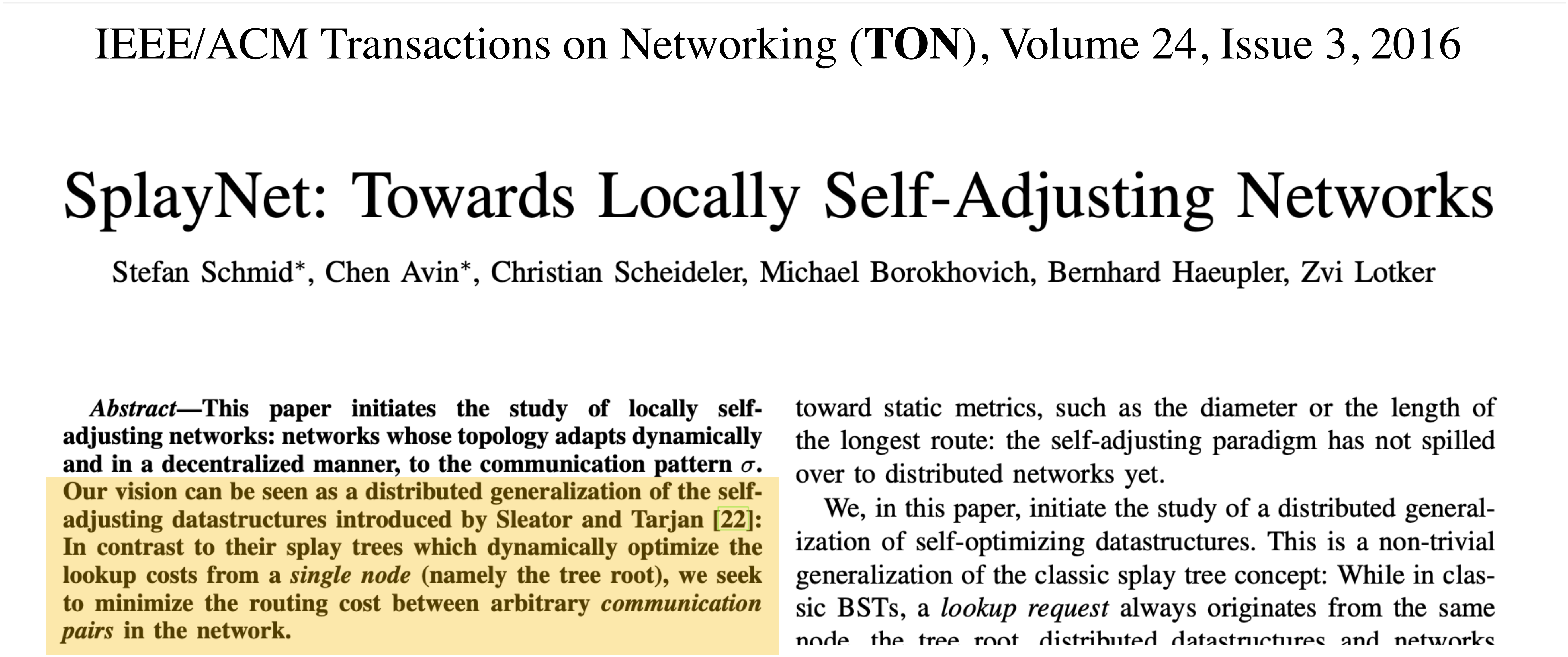
- The correctness proofs combine:
  - reconfiguration rules using local reasoning scale up via compositionality [Ahrens, Bozga, I, Katoen, OOPSLA'22]
  - havoc invariants using regular model checking techniques [Bozga, Bueri, I, CONCUR'22]
  - proving safety of assertions using parametric model checking techniques [Bozga, I, Sifakis, TCS' 23]

Architectures and Datastructures





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SplayNet: Towards Locally Self-Adjusting Networks

Stefan Schmid*, Chen Avin*, Christian Scheideler, Michael Borokhovich, Bernhard Haeupler, Zvi Lotker

***Abstract*—This paper initiates the study of locally self-adjusting networks: networks whose topology adapts dynamically and in a decentralized manner, to the communication pattern σ. Our vision can be seen as a distributed generalization of the self-adjusting datastructures introduced by Sleator and Tarjan [22]: In contrast to their splay trees which dynamically optimize the lookup costs from a *single node* (namely the tree root), we seek to minimize the routing cost between arbitrary *communication pairs* in the network.**

toward static metrics, such as the diameter or the length of the longest route: the self-adjusting paradigm has not spilled over to distributed networks yet.

We, in this paper, initiate the study of a distributed generalization of self-optimizing datastructures. This is a non-trivial generalization of the classic splay tree concept: While in classic BSTs, a *lookup request* always originates from the same node, the tree root, distributed datastructures and networks

- Network architectures are similar to the datastructures used in programming
- Used to design efficient routing algorithms that minimize internal traffic in datacenters
- We aim at proving correctness of self-adapting networks using a Configuration Logic (CL)

A Logic of Configurations (CL)

emp	the empty network

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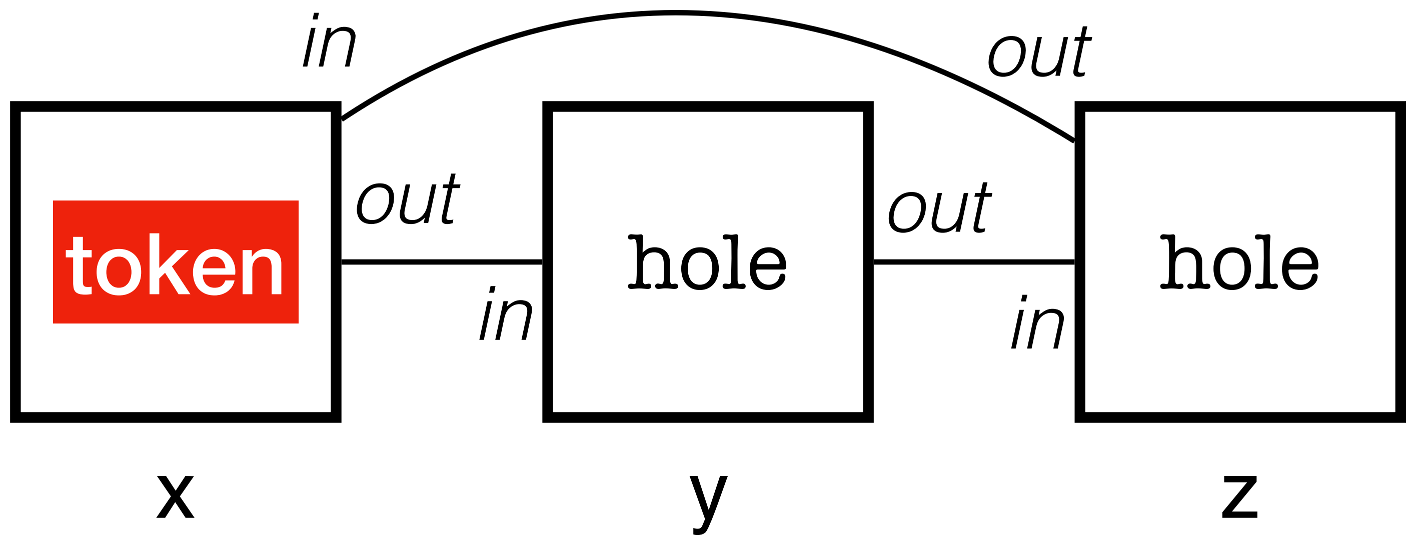

$[x]@token * \langle x.out,y.in\rangle * [y]@hole * \langle y.out,z.in\rangle * [z]@hole * \langle z.out, x.in\rangle$

A Logic of Configurations (CL)

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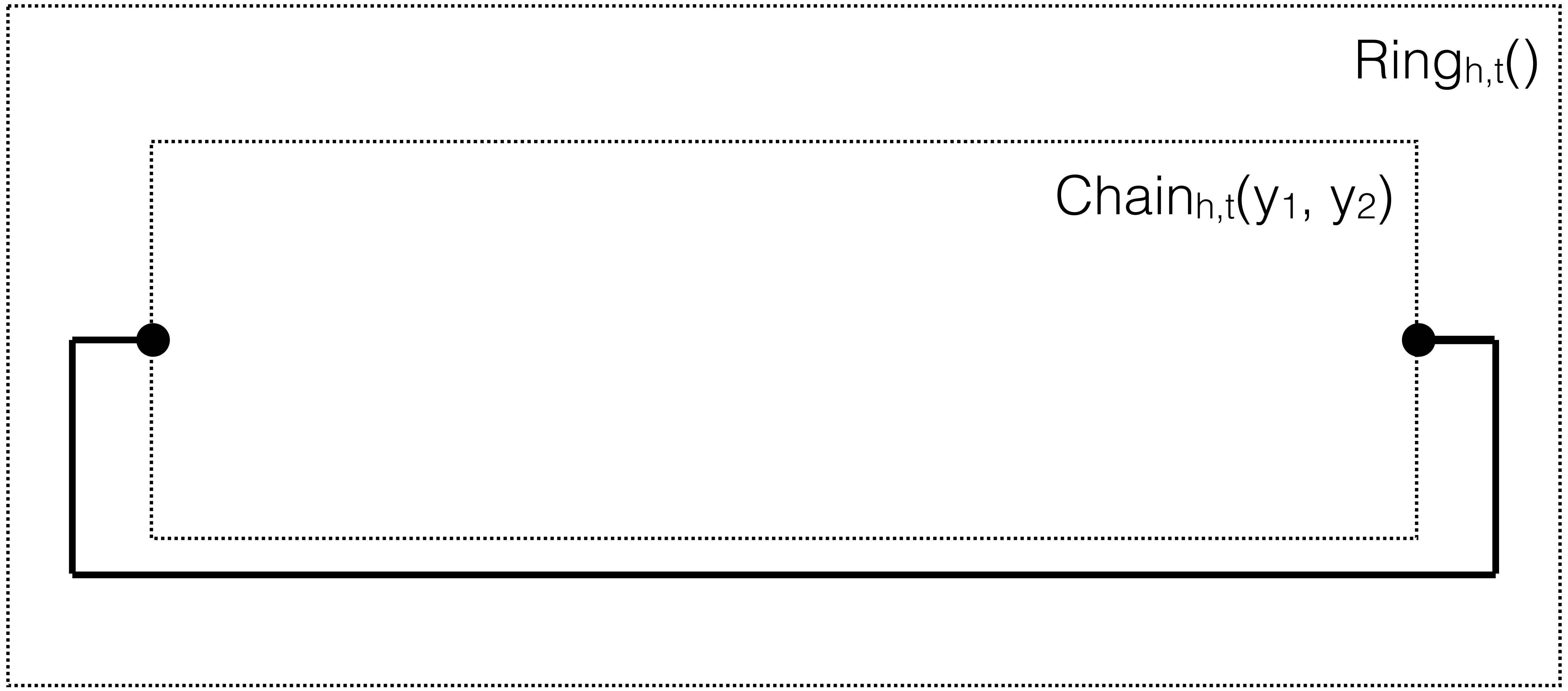
$[x]@q$	a single node in state q and no interactions
$\langle x_1.p_1 \text{ ...., } x_n.p_n\rangle$	a single interaction and no nodes
$\phi_1 * \phi_2$	separating conjunction (union of disjoint networks)
$\phi_1 \wedge \phi_2$	boolean conjunction
$\exists x \, . \, \phi$	existential quantification

Adding inductive definitions

$\mathrm{Ring}_{h,t}()$

Adding inductive definitions



$Ring_{h,t}() \leftarrow \exists y_1 \exists y_2 .\ Chain_{h,t}(y_1, y_2) * \langle y_2.out, y_1.in \rangle$

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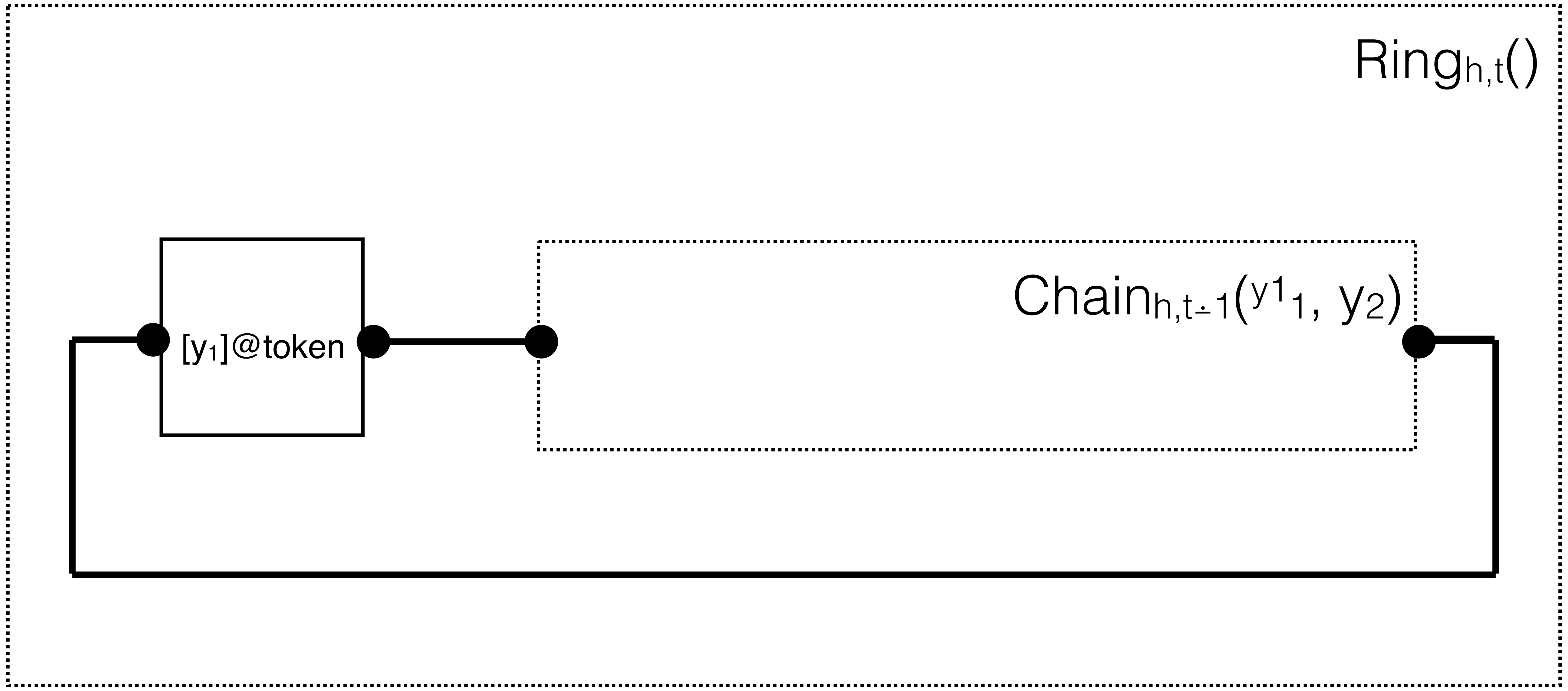

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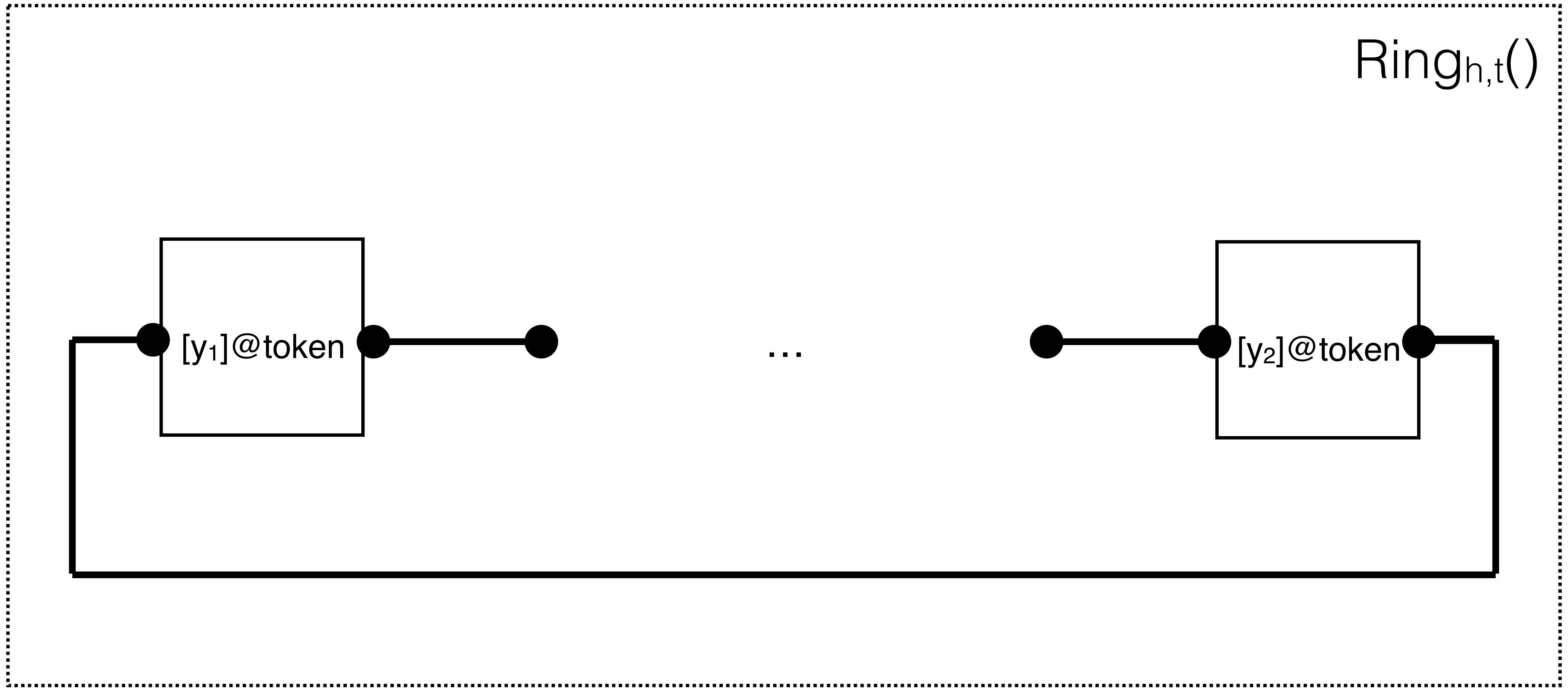
$\mathrm{Chain}_{h,t}(x, y) \leftarrow \exists z\ .\ [x]@\mathrm{token} * \langle x.\mathrm{out}, z.\mathrm{in}\rangle * \mathrm{Chain}_{h,t\dot{-}1}(z, y)$

$\mathrm{Chain}_{h,t}(x, y) \leftarrow \exists z\ .\ [x]@\mathrm{hole} * \langle x.\mathrm{out}, z.\mathrm{in}\rangle * \mathrm{Chain}_{h\dot{-}1,t}(z, y),\ n\dot{-}1 \overset{\mathrm{def}}{=} \max(0,n\text{-}1)$

Adding inductive definitions



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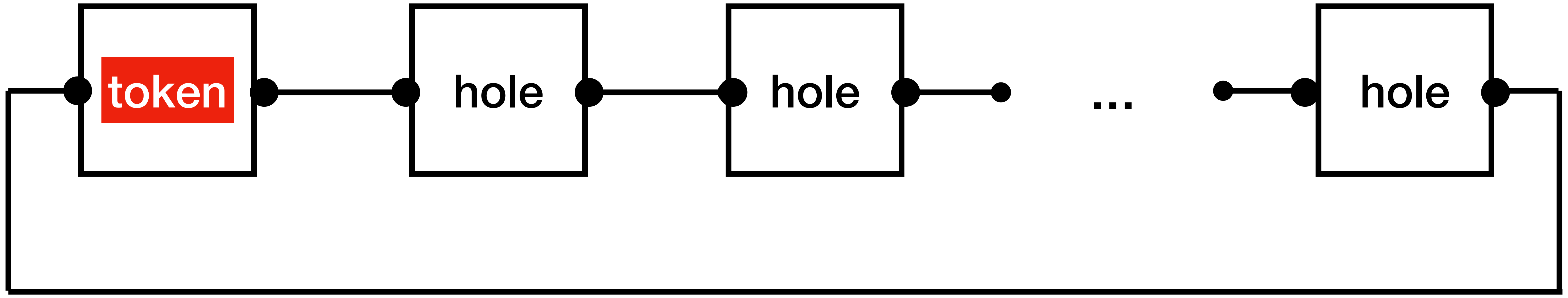
$\mathrm{Chain}_{h,t}(x, y) \leftarrow \exists z .\ [x]@\mathrm{hole} * \langle x.\mathrm{out}, z.\mathrm{in}\rangle * \mathrm{Chain}_{h\dot{-}1,t}(z, y),\ \ n\dot{-}1 \overset{\mathrm{def}}{=} \max(0,n-1)$

$\mathrm{Chain}_{0,1}(x,y) \leftarrow [x]@\mathrm{token} * x=y$ $\qquad\qquad$ $\mathrm{Chain}_{1,0}(x,y) \leftarrow [x]@\mathrm{hole} * x=y$

Programmed reconfigurability

- Sequential programming language based on:
  - primitives: new(x,q), delete(x), connect($x_1.p_1$, ..., $x_n.p_n$), disconnect($x_1.p_1$, ..., $x_n.p_n$)
  - conditional: with x_1, ..., x_n : ϕ do R od, where ϕ is a CL formula with no predicates
  - sequential composition (R_1; R_2), iteration (R^*) and nondeterministic choice ($R_1 + R_2$)

An example: token-ring node removal



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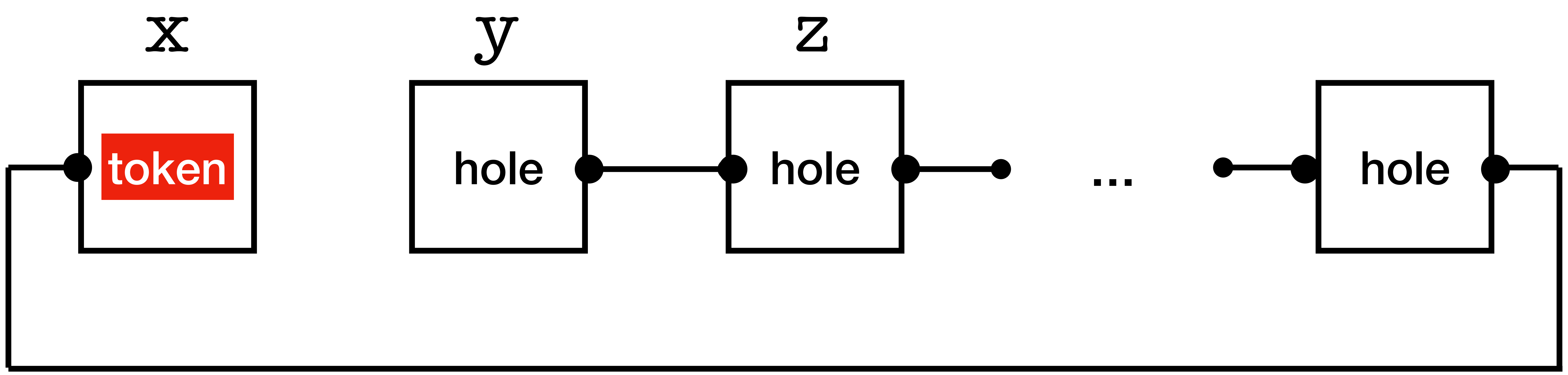

with x,y,z : ⟨x.out,y.in⟩ * [y]@hole * ⟨y.out,z.in⟩ do
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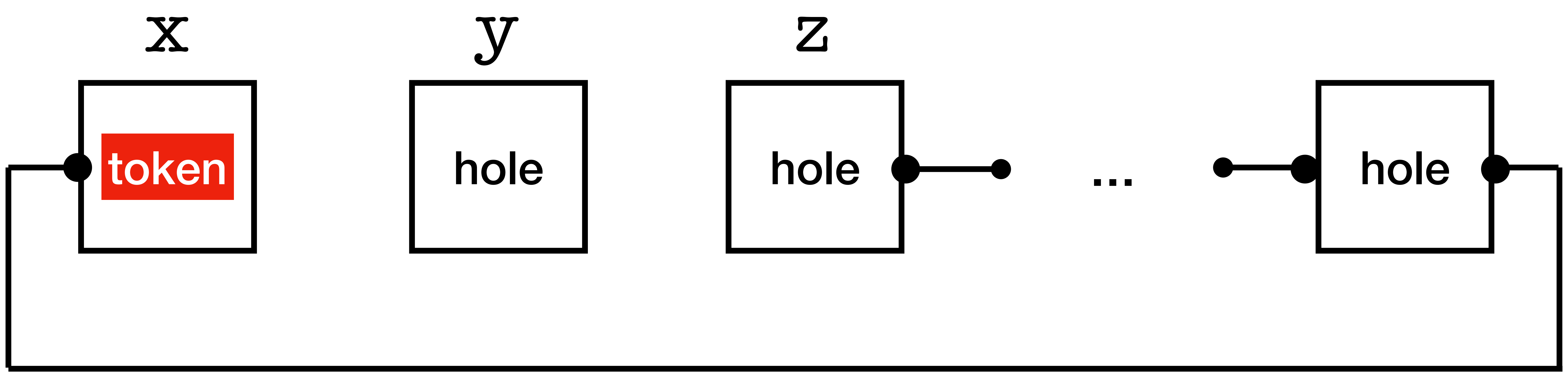

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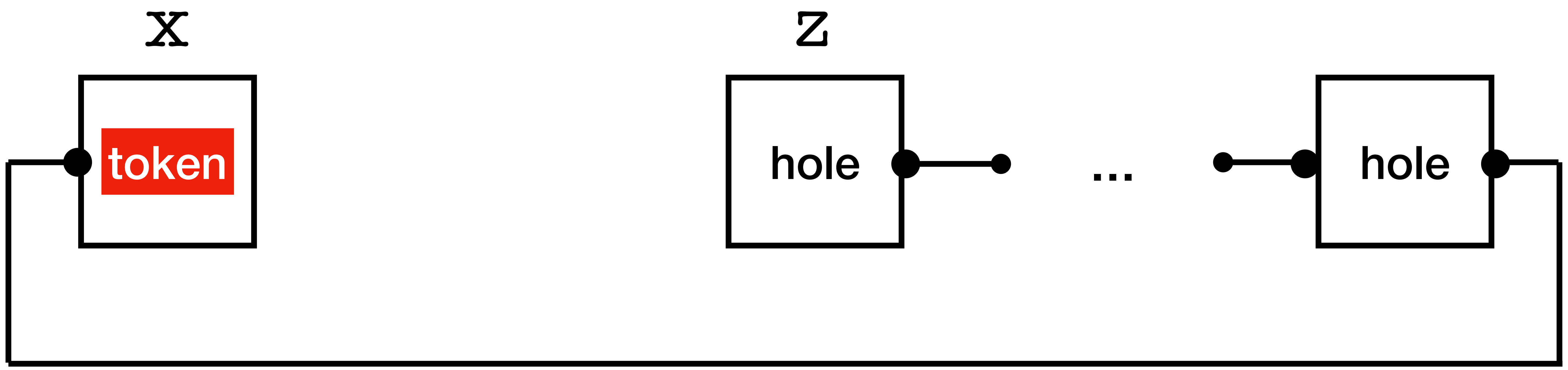

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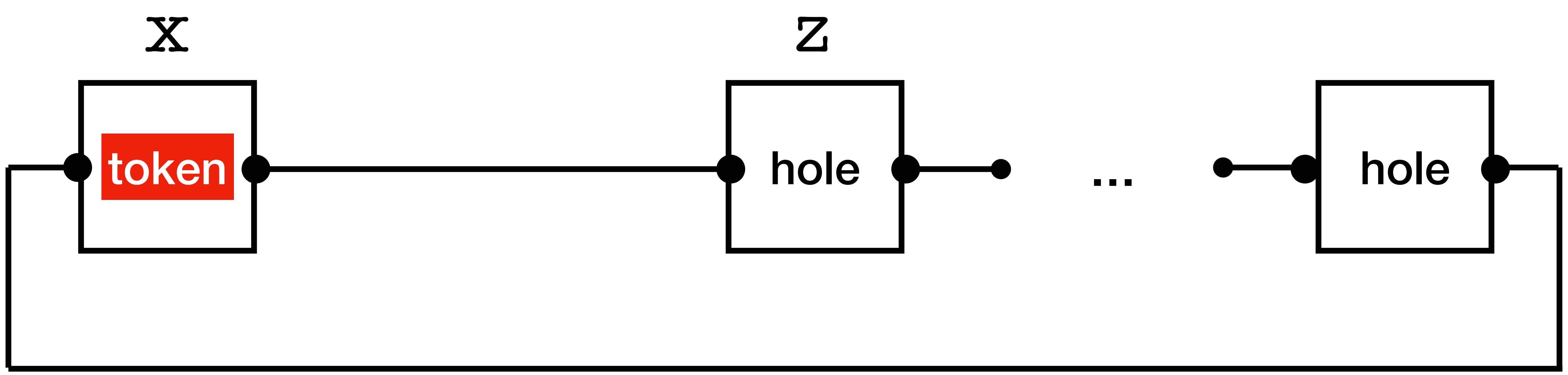

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od
```

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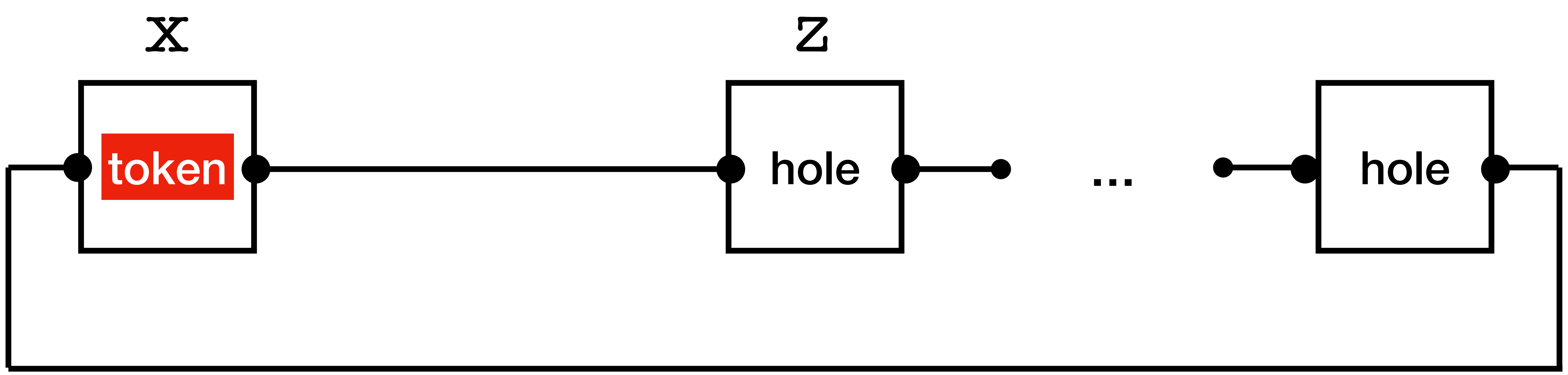

```
{ $Ring_{2,1}()$ }
  with x,y,z : ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ do
    disconnect(x.out,y.in);
    disconnect(y.out,z.in);
    delete(y);
    connect(x.out,z.in);
  od
{ $Ring_{1,1}()$ }
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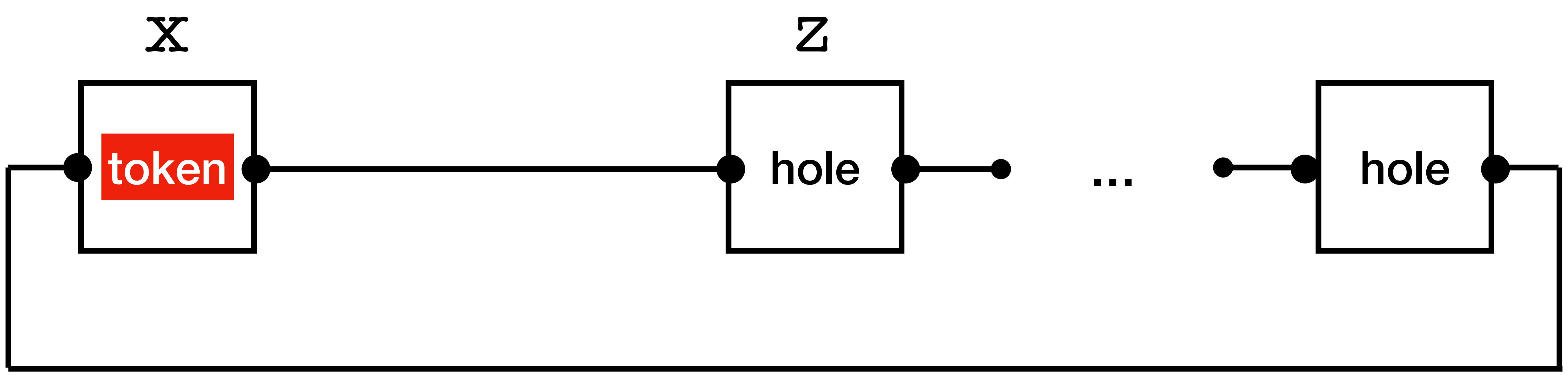
An example: token-ring node removal



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od
```

{ $\text{Ring}_{1,1}()$ } → safe

Local Reasoning

{emp} **new(x,q)** {[x]@q}

{[x]@q} **delete(x)** {emp}

{emp} **connect($x_1.p_1$, ..., $x_n.p_n$)** { ⟨$x_1.p_1$ …., $x_n.p_n$⟩ }

{ ⟨$x_1.p_1$ …., $x_n.p_n$⟩ } **disconnect($x_1.p_1$, ..., $x_n.p_n$)** {emp}

A local specification only mentions those resources that are necessary to avoid faulting

Local Reasoning

$\{\mathrm{emp}\}\ \mathbf{new}(\mathbf{x},\mathbf{q})\ \{[x]@q\}$

$\{[x]@q\}\ \mathbf{delete}(\mathbf{x})\ \{\mathrm{emp}\}$

$\{\mathrm{emp}\}\ \mathbf{connect}(\mathbf{x}_1.\mathbf{p}_1, \ldots, \mathbf{x}_n.\mathbf{p}_n)\ \{\ \langle x_1.p_1\ \ldots., x_n.p_n\rangle\ \}$

$\{\ \langle x_1.p_1\ \ldots., x_n.p_n\rangle\ \}\ \mathbf{disconnect}(\mathbf{x}_1.\mathbf{p}_1, \ldots, \mathbf{x}_n.\mathbf{p}_n)\ \{\mathrm{emp}\}$

$$\frac{\{\phi\}\ \mathrm{R}\ \{\Psi\}}{\{\phi\ {*}\ F\}\ \mathrm{R}\ \{\Psi\ {*}\ F\}}$$

if R is a local program and

$\mathrm{modifies}(\mathrm{R}) \cap \mathrm{fv}(F) = \varnothing$

A local specification only mentions those resources that are necessary to avoid faulting

The frame rule plugs a local specification into a global context

Which Reconfiguration Programs are Local?

Let Γ be the set of configurations

An action is a function $f : \Gamma \rightarrow \mathrm{pow}(\Gamma)^{T}$, where $\mathcal{S} \subseteq T$, $\forall \mathcal{S} \in \mathrm{pow}(\Gamma)$

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An action f is local ⇔ $f(\gamma_1 * \gamma_2) \subseteq f(\gamma_1) * \{\gamma_2\}$

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- $\mathtt{new(x,q)}$, $\mathtt{delete(x)}$, $\mathtt{connect(x_1.p_1, \ldots, x_n.p_n)}$, $\mathtt{disconnect(x_1.p_1, \ldots, x_n.p_n)}$
- with $\mathtt{x_1, \ldots, x_n} : \phi$ do … od, where ϕ is a conjunction of equalities
- nondeterministic choices $\mathtt{R_1} + \mathtt{R_2}$ between local programs

Which Reconfiguration Programs are Local?

Let Γ be the set of configurations

An action is a function $f : \Gamma \rightarrow pow(\Gamma)^{T}$, where $\mathcal{S} \subseteq T$, $\forall \mathcal{S} \in pow(\Gamma)$

An action f is local $\Leftrightarrow f(\gamma_1 * \gamma_2) \subseteq f(\gamma_1) * \{\gamma_2\}$

- new(x,q), delete(x), connect($x_1.p_1, \ldots, x_n.p_n$), disconnect($x_1.p_1, \ldots, x_n.p_n$)
- with $x_1, \ldots, x_n : \phi$ do … od, where ϕ is a conjunction of equalities
- nondeterministic choices $R_1 + R_2$ between local programs

Non-local programs:

- sequential compositions $R_1; R_2$
- with $x_1, \ldots, x_n : \phi$ do … od, where ϕ contains node/interaction atoms

Sequential Composition

$$\frac{\{\phi\}\ R_1\ \{\theta\} \qquad \{\theta\}\ R_2\ \{\Psi\}}{\{\phi\}\ R_1;\ R_2\ \{\Psi\}}$$

Sequential Composition

$$\frac{\{\phi\}\ R_1\ \{\theta\} \qquad \{\theta\}\ R_2\ \{\Psi\}}{\{\phi\}\ R_1;\ R_2\ \{\Psi\}}$$

θ is havoc invariant

A formula ϕ is havoc invariant ⇔ *for each model ɣ of ϕ and each state change ɣ →* ɣ' corresponding to firing one or more interactions enabled in ɣ, ɣ' is a model of ϕ*

Conditional Rule

$$\frac{\{\phi \wedge (\theta * \mathrm{true})\}\ \mathtt{R}\ \{\Psi\}}{\{\phi\}\ \mathtt{with}\ \mathbf{x}{:}\theta\ \mathtt{do}\ \mathtt{R}\ \mathtt{od}\ \{\exists \mathbf{x}\ .\ \Psi\}} \qquad \mathrm{fv}(\phi) \cap \mathbf{x} = \varnothing$$

The premiss introduces both boolean and separating conjunction

Conditional Rule

$$\frac{\{\theta * F\}\ \mathtt{R}\ \{\Psi\}}{\{\phi\}\ \mathtt{with}\ \mathbf{x}{:}\theta\ \mathtt{do}\ \mathtt{R}\ \mathtt{od}\ \{\exists \mathbf{x}\ .\ \Psi\}}\quad \mathrm{fv}(\phi) \cap \mathbf{x} = \varnothing$$

The premiss introduces both boolean and separating conjunction

The boolean conjunction can be eliminated by solving a frame inference problem:

Find the strongest formula (if one exists) F such that $\phi \models \theta * F$

Back to the proof

```
{ $\text{Ring}_{2,1}()$ }

with x,y,z : ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ do

  disconnect(x.out,y.in);

  disconnect(y.out,z.in);

  delete(y);

  connect(x.out,z.in);

od

{ $\text{Ring}_{1,1}()$ }
```

Back to the proof

```
{ $\mathrm{Ring}_{2,1}()$ }
{ $\mathrm{Chain}_{2,1}(x,z)$ *  ⟨z.out,x.in⟩  }
with x,y,z : ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ do

  disconnect(x.out,y.in);

  disconnect(y.out,z.in);

  delete(y);

  connect(x.out,z.in);

od

{ $\mathrm{Ring}_{1,1}()$ }
```

Back to the proof

```
{ $\mathrm{Ring}_{2,1}()$ }
{ $\mathrm{Chain}_{2,1}(x,z)$ * ⟨z.out,x.in⟩ }
with x,y,z : ⟨x.out,y.in⟩ * [y]@hole * ⟨y.out,z.in⟩ do
{ ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ * $\mathrm{Chain}_{1,1}(z,x)$ }
  disconnect(x.out,y.in);

  disconnect(y.out,z.in);

  delete(y);

  connect(x.out,z.in);

od

{ $\mathrm{Ring}_{1,1}()$ }
```

Back to the proof

```
{ Ring$_{2,1}$() }
{ Chain$_{2,1}$(x,z) * ⟨z.out,x.in⟩ }
with x,y,z : ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ do
{ ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ * Chain$_{1,1}$(z,x) }
   disconnect(x.out,y.in);
{ [y]@hole* ⟨y.out,z.in⟩ * Chain$_{1,1}$(z,x) }
   disconnect(y.out,z.in);

   delete(y);

   connect(x.out,z.in);

od

{ Ring$_{1,1}$() }
```

{ ⟨x.out,y.in⟩ } disconnect(x.out,y.in) { emp }

Back to the proof

```
{ $Ring_{2,1}()$ }
{ $Chain_{2,1}(x,z)$ * ⟨z.out,x.in⟩ }
with x,y,z : ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ do
{ ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ * $Chain_{1,1}(z,x)$ }
  disconnect(x.out,y.in);
{ [y]@hole * ⟨y.out,z.in⟩ * $Chain_{1,1}(z,x)$ }
  disconnect(y.out,z.in);

  delete(y);

  connect(x.out,z.in);

od

{ $Ring_{1,1}()$ }
```

{ ⟨x.out,y.in⟩ } disconnect(x.out,y.in) { emp }

{ ⟨y.out,z.in⟩ } disconnect(y.out,z.in) { emp }

Back to the proof

```
{ $\mathrm{Ring}_{2,1}()$ }
{ $\mathrm{Chain}_{2,1}(x,z)$ * ⟨z.out,x.in⟩ }
with x,y,z : ⟨x.out,y.in⟩ * [y]@hole * ⟨y.out,z.in⟩ do
{ ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ * $\mathrm{Chain}_{1,1}(z,x)$ }
  disconnect(x.out,y.in);
{ [y]@hole * ⟨y.out,z.in⟩ * $\mathrm{Chain}_{1,1}(z,x)$ }
  disconnect(y.out,z.in);
{ [y]@hole * $\mathrm{Chain}_{1,1}(z,x)$ }
  delete(y);

  connect(x.out,z.in);

od

{ $\mathrm{Ring}_{1,1}()$ }
```

{ ⟨x.out,y.in⟩ } disconnect(x.out,y.in) { emp }

{ ⟨y.out,z.in⟩ } disconnect(y.out,z.in) { emp }

Back to the proof

```
{ Ring$_{2,1}$() }
{ Chain$_{2,1}$(x,z) * ⟨z.out,x.in⟩ }
with x,y,z : ⟨x.out,y.in⟩ * [y]@hole * ⟨y.out,z.in⟩ do
{ ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ * Chain$_{1,1}$(z,x) }
  disconnect(x.out,y.in);
{ [y]@hole * ⟨y.out,z.in⟩ * Chain$_{1,1}$(z,x) }
  disconnect(y.out,z.in);
{ [y]@hole * Chain$_{1,1}$(z,x) }
  delete(y);

  connect(x.out,z.in);

od

{ Ring$_{1,1}$() }
```

{ ⟨x.out,y.in⟩ } disconnect(x.out,y.in) { emp }

{ ⟨y.out,z.in⟩ } disconnect(y.out,z.in) { emp }

Back to the proof

{ Ring$_{2,1}$() }

{ Chain$_{2,1}$(x,z) * ⟨z.out,x.in⟩ }

with x,y,z : ⟨x.out,y.in⟩ * [y]@hole * ⟨y.out,z.in⟩ do

{ ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ * Chain$_{1,1}$(z,x) }

  disconnect(x.out,y.in);

{ [y]@hole * ⟨y.out,z.in⟩ * Chain$_{1,1}$(z,x) }

  disconnect(y.out,z.in);

{ [y]@hole * Chain$_{1,1}$(z,x) }

  delete(y);

{ Chain$_{1,1}$(z,x) }

  connect(x.out,z.in);

od

{ Ring$_{1,1}$() }

{ ⟨x.out,y.in⟩ } disconnect(x.out,y.in) { emp }

{ ⟨y.out,z.in⟩ } disconnect(y.out,z.in) { emp }

{ [y] } delete(y) { emp }

Back to the proof

```
{ $\mathrm{Ring}_{2,1}()$ }
{ $\mathrm{Chain}_{2,1}(x,z) * \langle z.out,x.in \rangle$ }
with x,y,z : $\langle x.out,y.in \rangle * [y]@hole * \langle y.out,z.in \rangle$ do
{ $\langle x.out,y.in \rangle * [y]@hole * \langle y.out,z.in \rangle * \mathrm{Chain}_{1,1}(z,x)$ }
   disconnect(x.out,y.in);
{ $[y]@hole * \langle y.out,z.in \rangle * \mathrm{Chain}_{1,1}(z,x)$ }
   disconnect(y.out,z.in);
{ $[y]@hole * \mathrm{Chain}_{1,1}(z,x)$ }
   delete(y);
{ $\mathrm{Chain}_{1,1}(z,x)$ }
   connect(x.out,z.in);
{ $\mathrm{Chain}_{1,1}(z,x) * \langle x.out,z.in \rangle$ }
od
{ $\exists x \exists z.\mathrm{Chain}_{1,1}(z,x) * \langle z.out,x.in \rangle$ }
{ $\mathrm{Ring}_{1,1}()$ }
```

{ $\langle x.out,y.in \rangle$ } disconnect(x.out,y.in) { emp }

{ $\langle y.out,z.in \rangle$ } disconnect(y.out,z.in) { emp }

{ [y] } delete(y) { emp }

{ emp } connect(x.out,z.in) { $\langle x.out,z.in \rangle$ }

Back to the proof

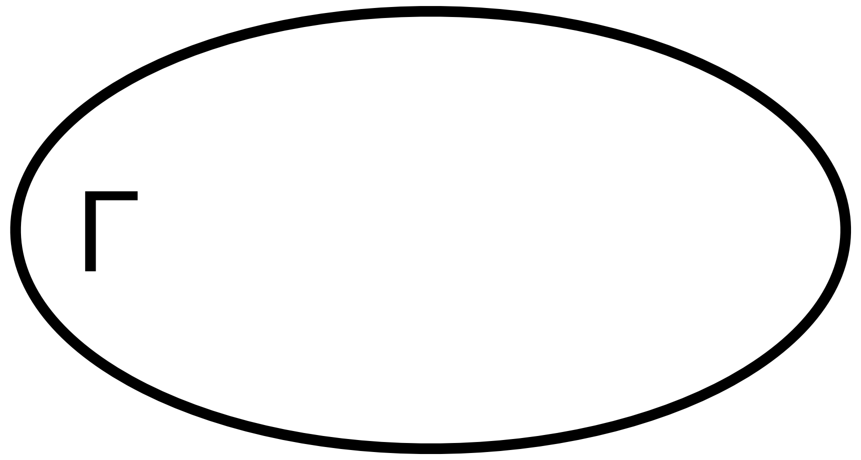
```
{ $Ring_{2,1}()$ }
{ $Chain_{2,1}(x,z)$ * ⟨z.out,x.in⟩ }
with x,y,z : ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ do
{ ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ * $Chain_{1,1}(z,x)$ }
   disconnect(x.out,y.in);
{ [y]@hole * ⟨y.out,z.in⟩ * $Chain_{1,1}(z,x)$ }
   disconnect(y.out,z.in);
{ [y]@hole * $Chain_{1,1}(z,x)$ }
   delete(y);
{ $Chain_{1,1}(z,x)$ }
   connect(x.out,z.in);
{ $Chain_{1,1}(z,x)$ * ⟨x.out,z.in⟩ }
od
{ ∃x∃z.$Chain_{1,1}(z,x)$ * ⟨z.out,x.in⟩ }
{ $Ring_{1,1}()$ }
```

Back to the proof

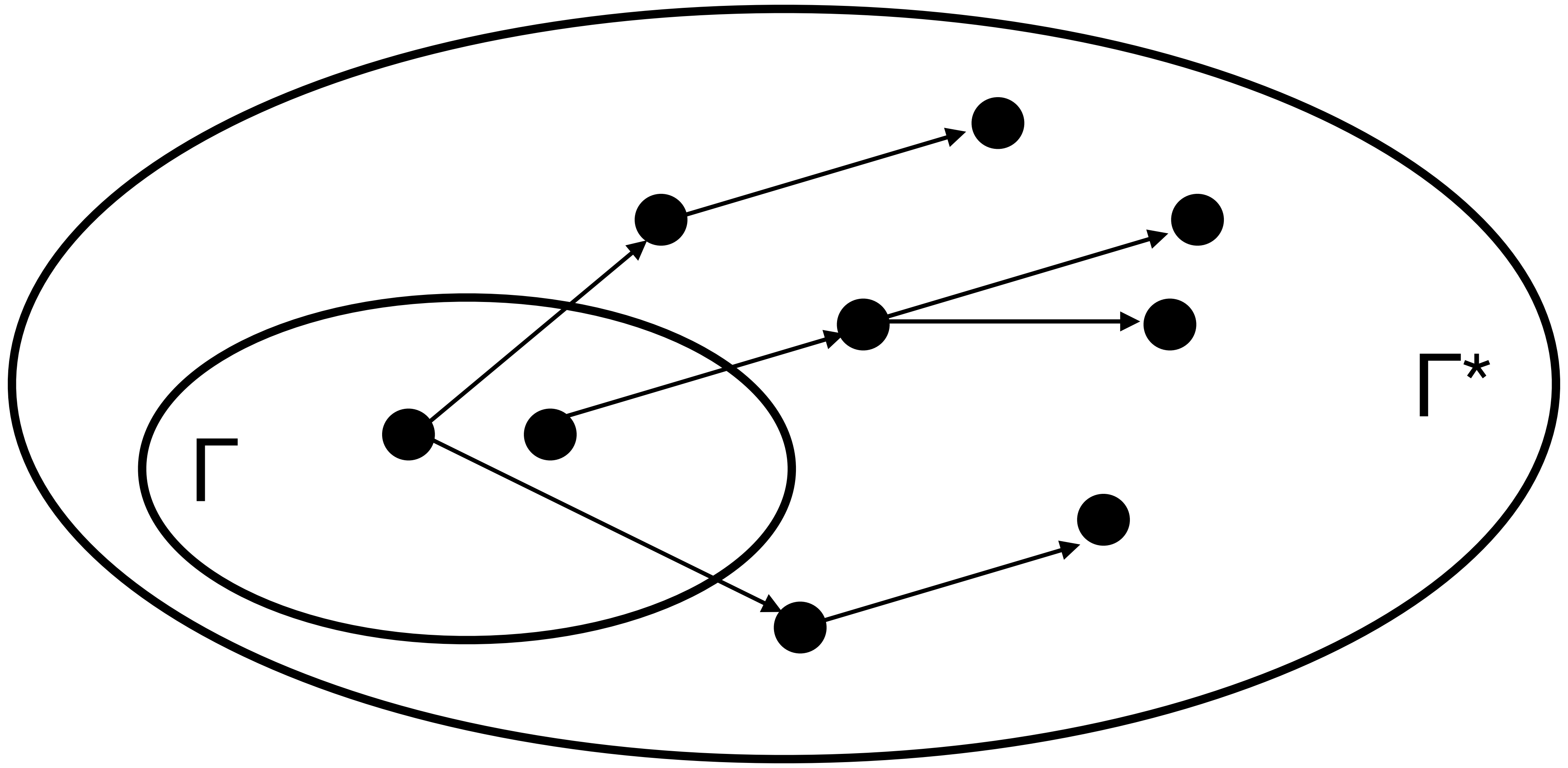
```
{ $\text{Ring}_{2,1}()$ }
{ $\text{Chain}_{2,1}(x,z)$ * ⟨z.out,x.in⟩ }
with x,y,z : ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ do
{ ⟨x.out,y.in⟩ * [y]@hole* ⟨y.out,z.in⟩ * $\text{Chain}_{1,1}(z,x)$ }
   disconnect(x.out,y.in);
{ [y]@hole * ⟨y.out,z.in⟩ * $\text{Chain}_{1,1}(z,x)$ }
   disconnect(y.out,z.in);
{ [y]@hole * $\text{Chain}_{1,1}(z,x)$ }
   delete(y);
{ $\text{Chain}_{1,1}(z,x)$ }
   connect(x.out,z.in);
{ $\text{Chain}_{1,1}(z,x)$ * ⟨x.out,z.in⟩ }
od
{ $\exists x \exists z.\text{Chain}_{1,1}(z,x)$ * ⟨z.out,x.in⟩ }
{ $\text{Ring}_{1,1}()$ }
```

havoc invariant?

Checking Havoc Invariance

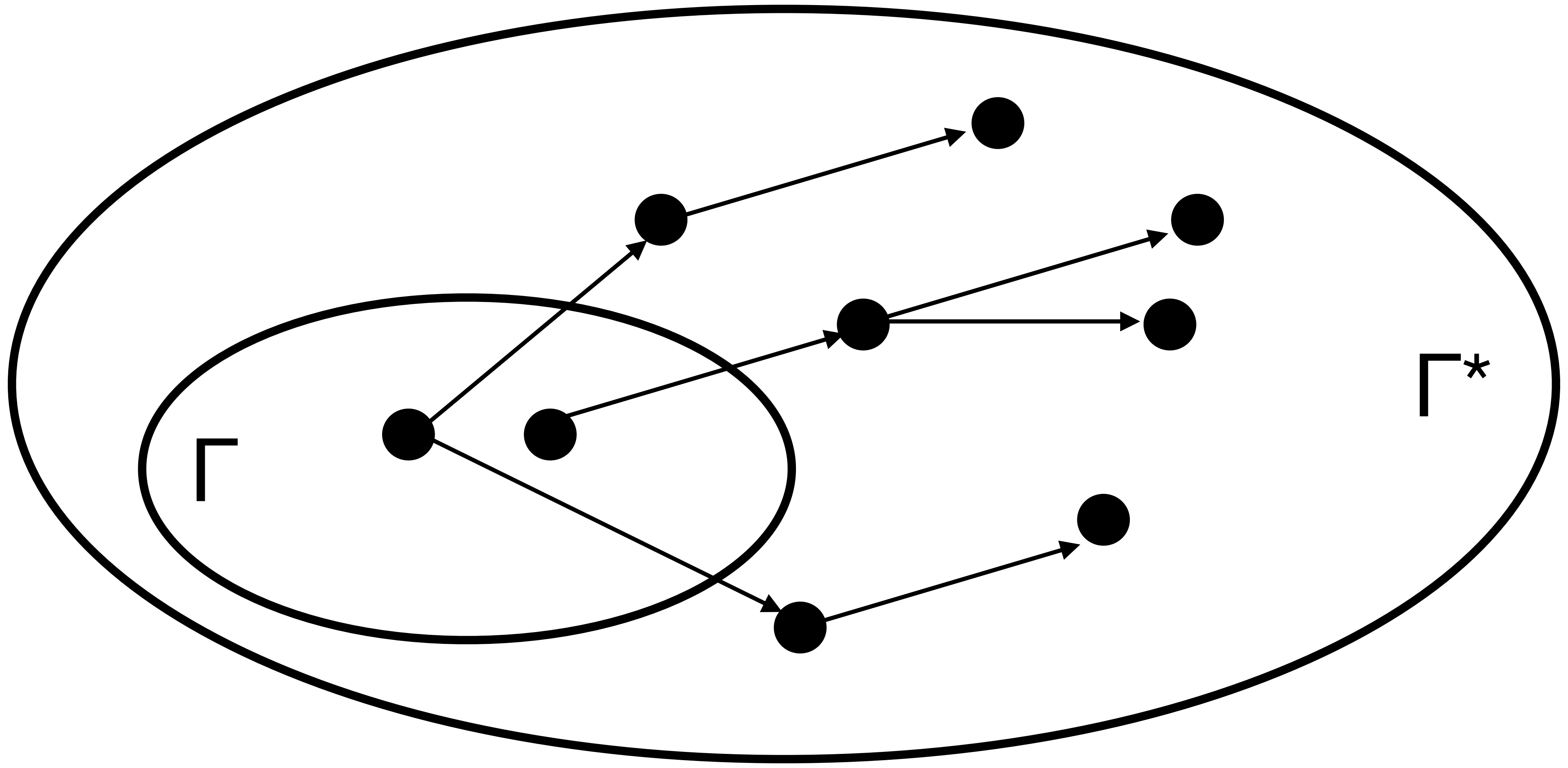


Checking Havoc Invariance

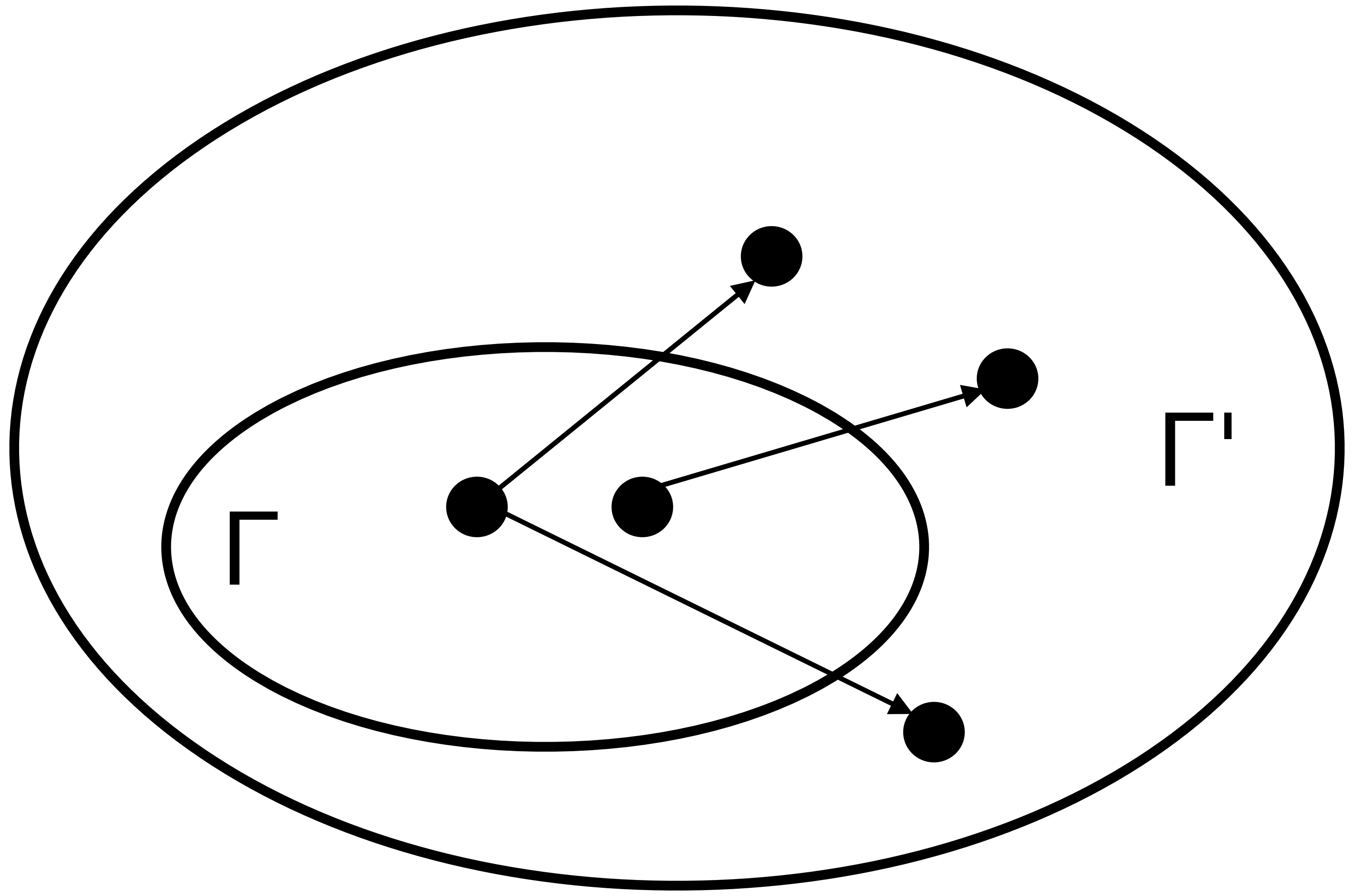


Γ* ⊆ Γ ?

Checking Havoc Invariance



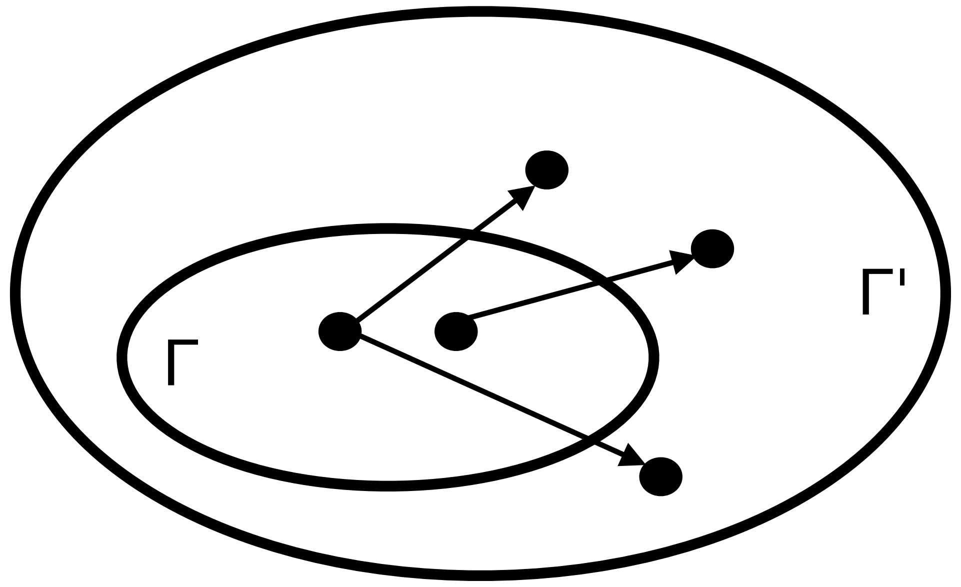
Γ* ⊆ Γ ?



Γ' ⊆ Γ ?

Checking Havoc Invariance

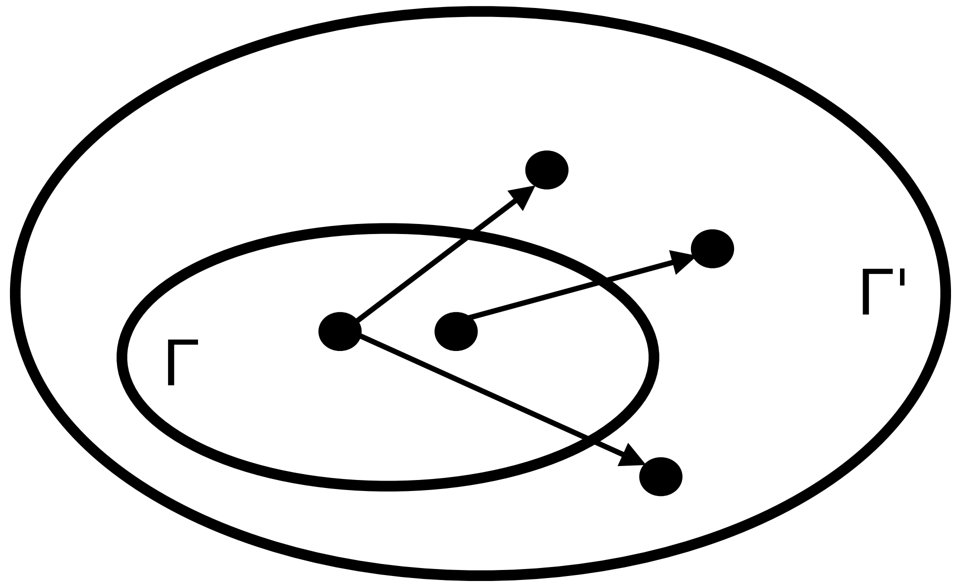
(Δ,A) describes Γ ⇢ (Δ',A') describes Γ'



$\Gamma' \subseteq \Gamma$?

Checking Havoc Invariance

(Δ,A) describes Γ ⋯⋯→ (Δ',A') describes Γ'



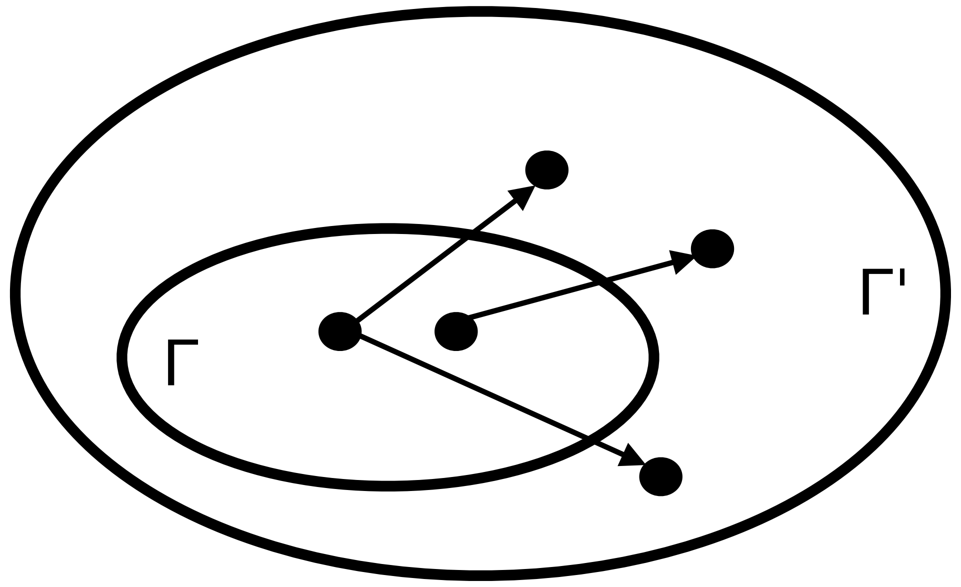
Γ' ⊆ Γ ?

Configurations are encoded as unfolding trees labeled with CL formulae

Checking Havoc Invariance

(Δ,A) describes Γ

(Δ',A') describes Γ'



Γ' ⊆ Γ ?

$\mathscr{A}_{\Delta,A}$ tree automaton

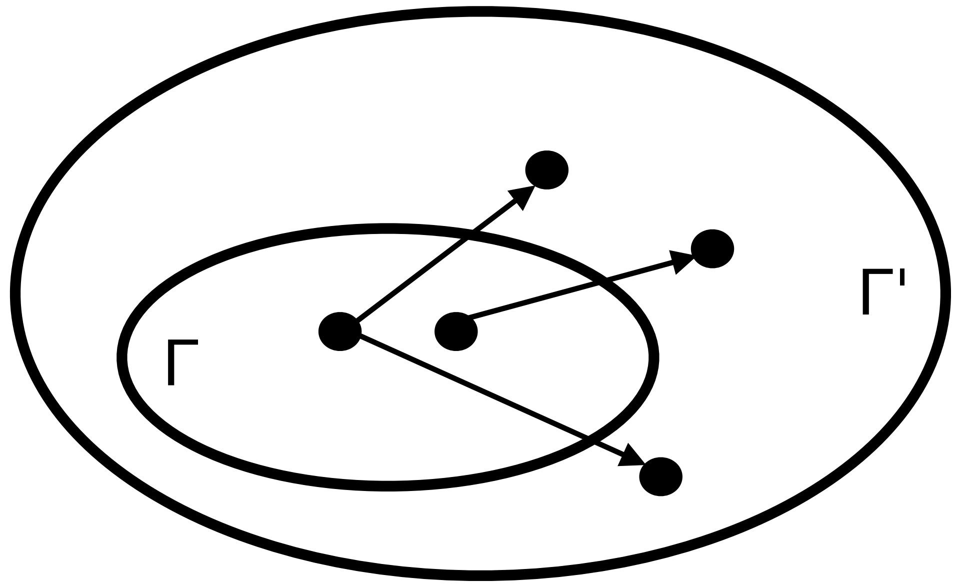
recognizing the unfolding trees of Δ for the formula $A(x_1 \ldots x_n)$

Configurations are encoded as unfolding trees labeled with CL formulae

Checking Havoc Invariance

(Δ,A) describes Γ

(Δ',A') describes Γ'



Γ' ⊆ Γ ?

$\mathcal{A}_{\Delta,A}$ tree automaton

recognizing the unfolding trees of Δ for the formula $A(x_1 \ldots x_n)$

$\mathcal{T}$ tree transducer

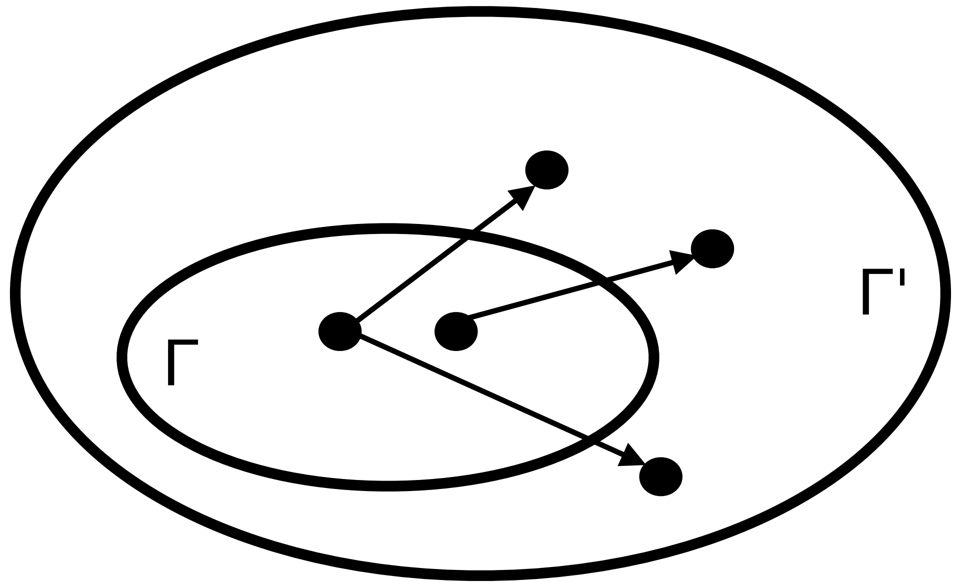
encoding the effect of one havoc step

Configurations are encoded as unfolding trees labeled with CL formulae

Checking Havoc Invariance

(Δ,A) describes Γ

(Δ',A') describes Γ'



Γ' ⊆ Γ ?

$\mathscr{A}_{\Delta,A}$ tree automaton

recognizing the unfolding trees of Δ for the formula A(x_1 ... x_n)

$\mathscr{T}$ tree transducer

encoding the effect of one havoc step

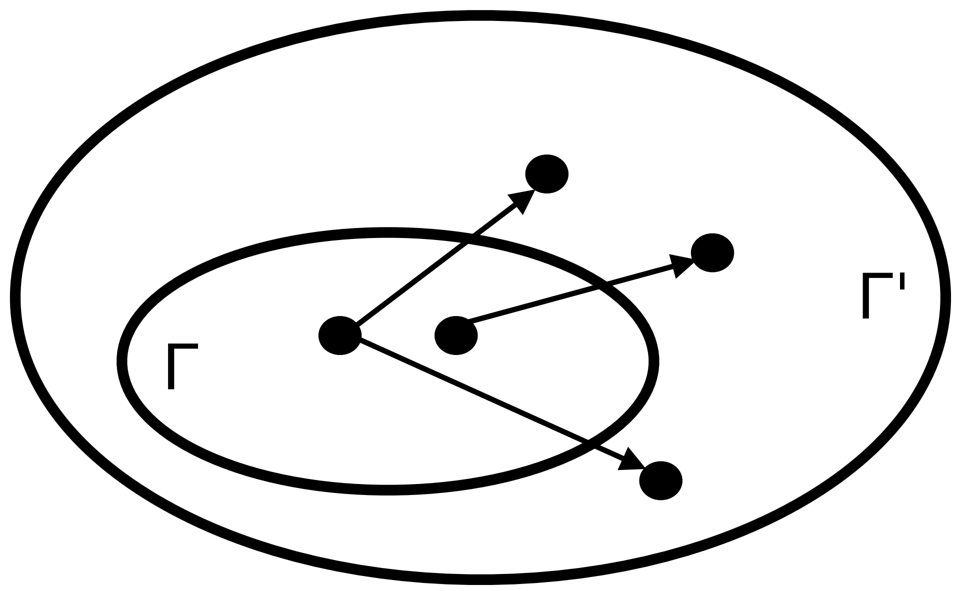
$\mathscr{T}$ ($\mathscr{A}_{\Delta,A}$)

Configurations are encoded as unfolding trees labeled with CL formulae

Checking Havoc Invariance

(Δ,A) describes Γ

(Δ',A') describes Γ'



Γ' ⊆ Γ ?

$\mathcal{A}_{\Delta,A}$ tree automaton

recognizing the unfolding trees of Δ for the formula $A(x_1 \ldots x_n)$

$\mathcal{T}$ tree transducer

encoding the effect of one havoc step

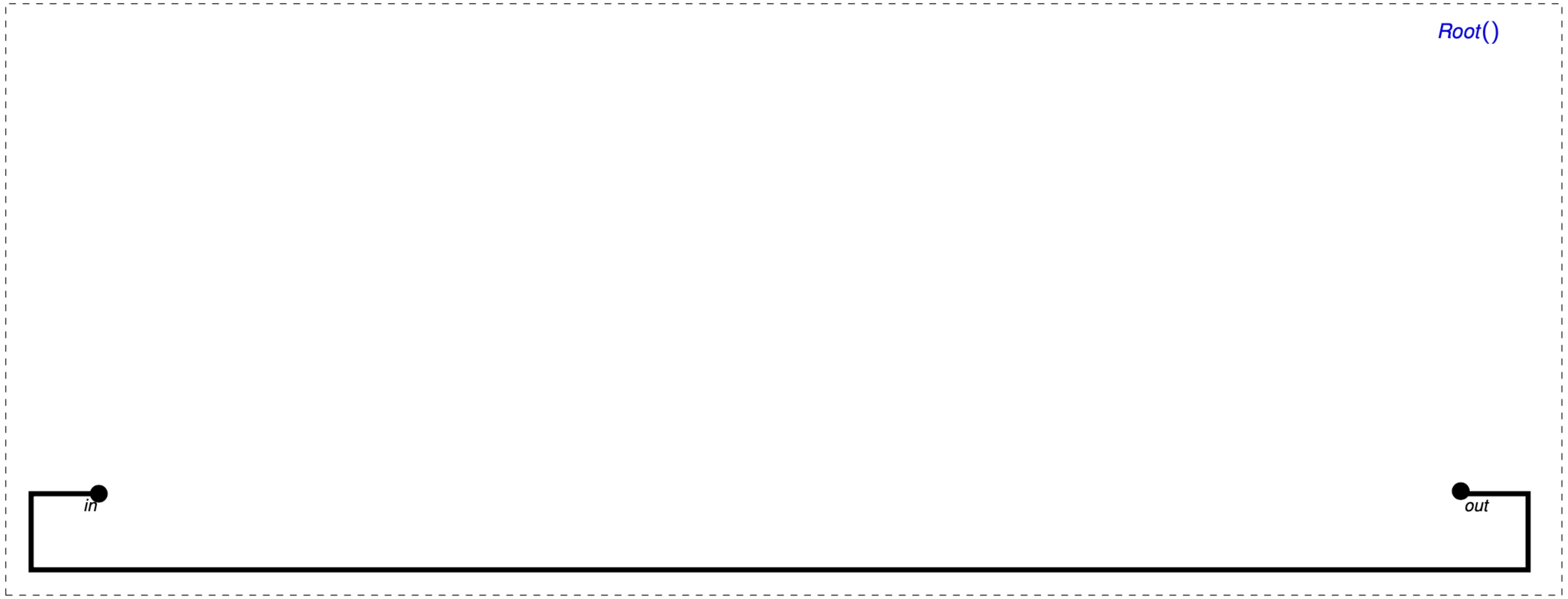
$\mathcal{T}(\mathcal{A}_{\Delta,A})$

Configurations are encoded as unfolding trees labeled with CL formulae

Check the entailment $A'(x_1 \ldots x_n) \models_{\Delta \cup \Delta'} A(x_1 \ldots x_n)$

A Tree with Leaves Linked in a Ring

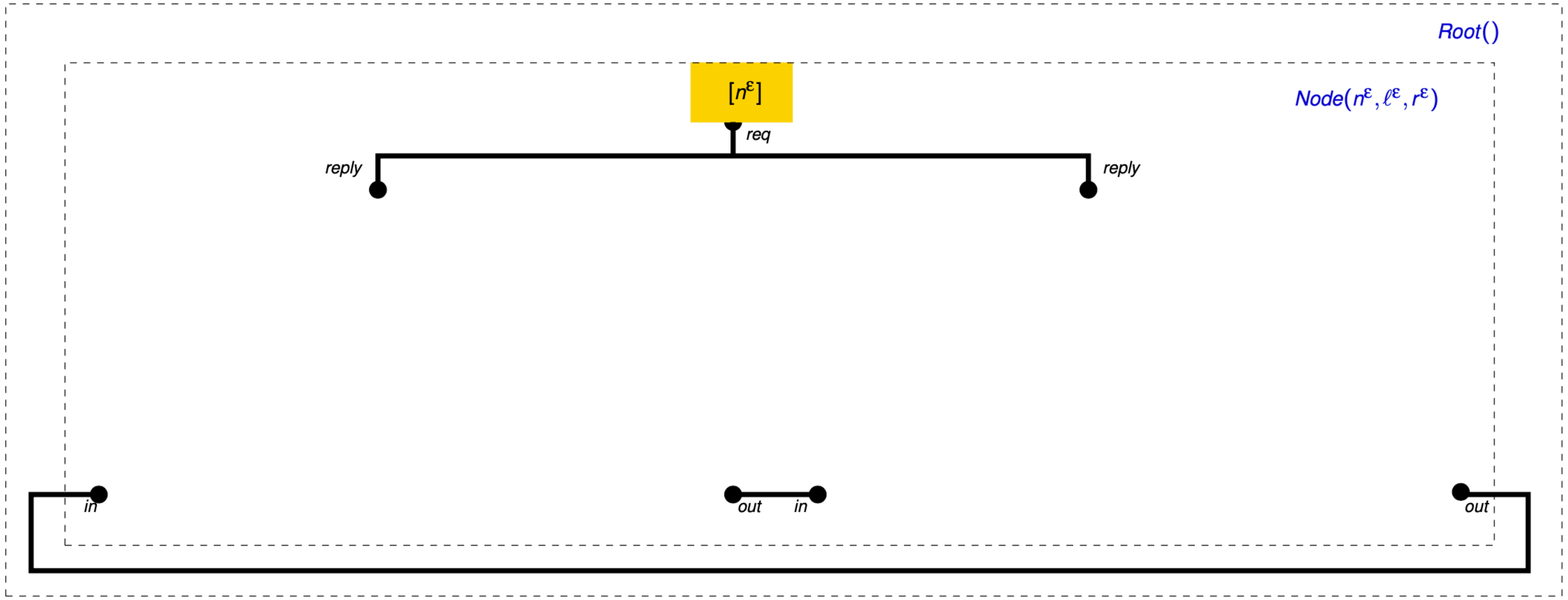
(α) $\mathit{Root}() \leftarrow \exists n \exists \ell \exists r\ .\ \langle r.\mathit{out}, \ell.\mathit{in} \rangle * \mathit{Node}(n, \ell, r)$



A Tree with Leaves Linked in a Ring

$$(\alpha) \quad \mathit{Root}() \leftarrow \exists n \exists \ell \exists r \,.\, \langle r.\mathit{out}, \ell.\mathit{in} \rangle * \mathit{Node}(n, \ell, r)$$

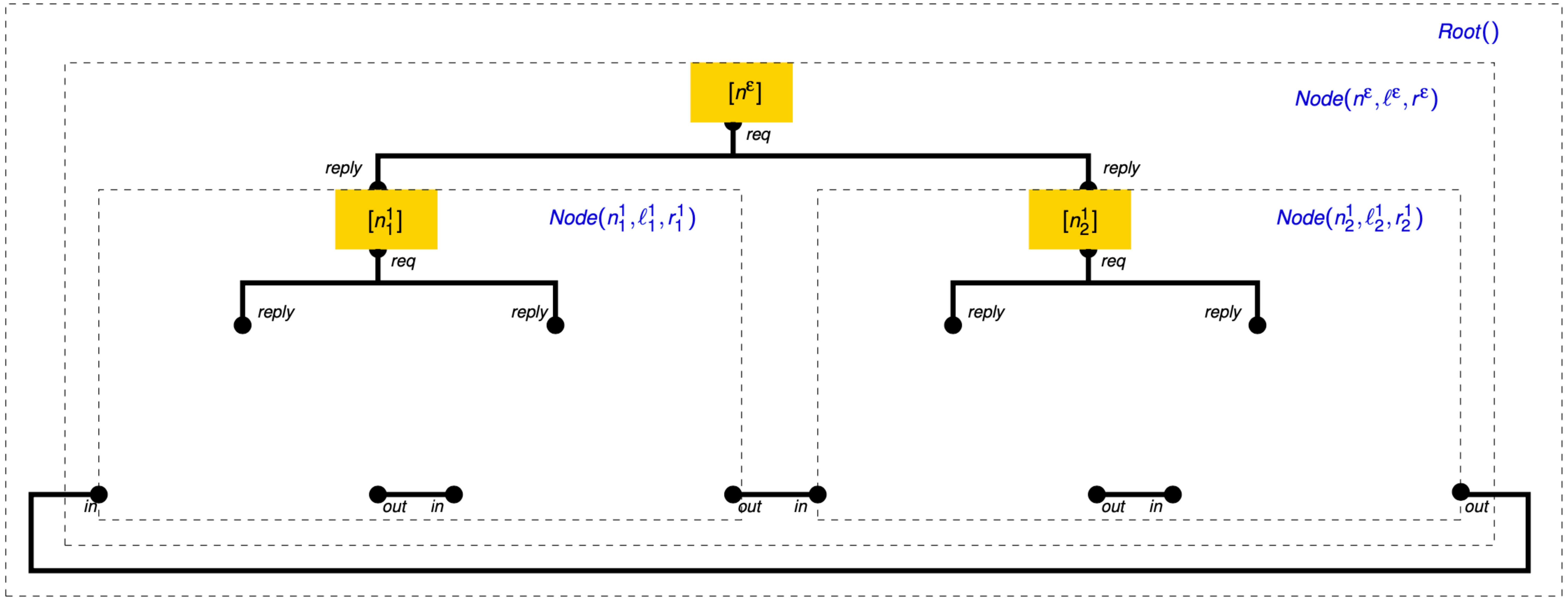
$$(\beta) \quad \mathit{Node}(n, \ell, r) \leftarrow \exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [n] * \langle n.\mathit{req}, n_1.\mathit{reply}, n_2.\mathit{reply} \rangle * \langle r_1.\mathit{out}, \ell_2.\mathit{in} \rangle * \mathit{Node}(n_1, \ell, r_1) * \mathit{Node}(n_2, \ell_2, r)$$



A Tree with Leaves Linked in a Ring

$$(\alpha)\quad \mathit{Root}() \leftarrow \exists n \exists \ell \exists r\ .\ \langle r.\mathit{out}, \ell.\mathit{in}\rangle * \mathit{Node}(n,\ell,r)$$

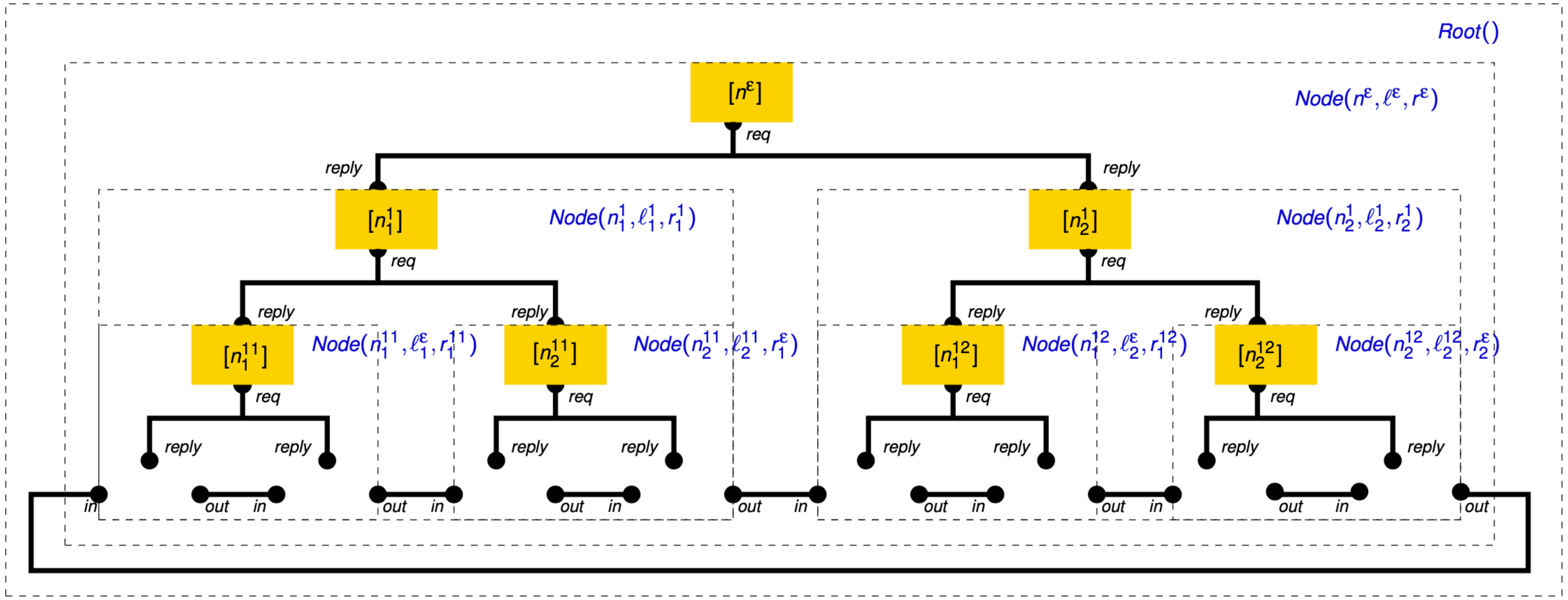
$$(\beta)\quad \mathit{Node}(n,\ell,r) \leftarrow \exists n_1 \exists r_1 \exists n_2 \exists \ell_2\ .\ [n] * \langle n.\mathit{req}, n_1.\mathit{reply}, n_2.\mathit{reply}\rangle * \langle r_1.\mathit{out}, \ell_2.\mathit{in}\rangle * \mathit{Node}(n_1,\ell,r_1) * \mathit{Node}(n_2,\ell_2,r)$$



$\mathit{Root}()$
$[n^{\varepsilon}]$
req
$\mathit{Node}(n^{\varepsilon},\ell^{\varepsilon},r^{\varepsilon})$
reply
reply
$[n_1^1]$
$\mathit{Node}(n_1^1,\ell_1^1,r_1^1)$
req
$[n_2^1]$
$\mathit{Node}(n_2^1,\ell_2^1,r_2^1)$
req
reply
reply
reply
reply
in
out
in
out
in
out
in
out

A Tree with Leaves Linked in a Ring

$$
\begin{array}{lrl}
(\alpha) & Root() & \leftarrow \exists n \exists \ell \exists r \,.\, \langle r.out, \ell.in \rangle * Node(n, \ell, r) \\
(\beta) & Node(n, \ell, r) & \leftarrow \exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [n] * \langle n.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle * Node(n_1, \ell, r_1) * Node(n_2, \ell_2, r)
\end{array}
$$



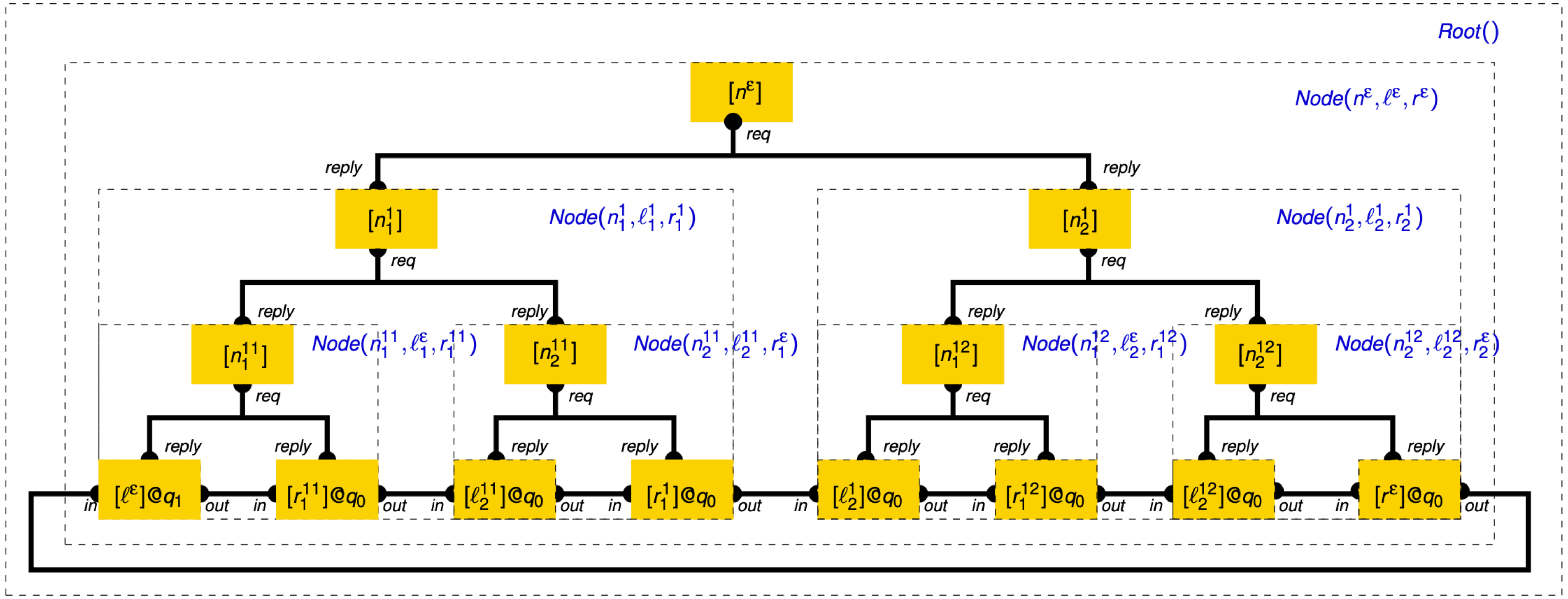
Root()
$[n^\varepsilon]$
req
$Node(n^\varepsilon, \ell^\varepsilon, r^\varepsilon)$
reply
reply
$[n_1^1]$
req
$Node(n_1^1, \ell_1^1, r_1^1)$
$[n_2^1]$
req
$Node(n_2^1, \ell_2^1, r_2^1)$
reply
reply
reply
reply
$[n_1^{11}]$
$Node(n_1^{11}, \ell_1^\varepsilon, r_1^{11})$
$[n_2^{11}]$
$Node(n_2^{11}, \ell_2^{11}, r_1^\varepsilon)$
$[n_1^{12}]$
$Node(n_1^{12}, \ell_2^\varepsilon, r_1^{12})$
$[n_2^{12}]$
$Node(n_2^{12}, \ell_2^{12}, r_2^\varepsilon)$
req
req
req
req
reply
reply
reply
reply
reply
reply
reply
reply
in
out
in
out
in
out
in
out
in
out
in
out
in
out

A Tree with Leaves Linked in a Ring

$$(\alpha)\quad Root() \leftarrow \exists n \exists \ell \exists r\, .\, \langle r.out, \ell.in \rangle * Node(n,\ell,r)$$

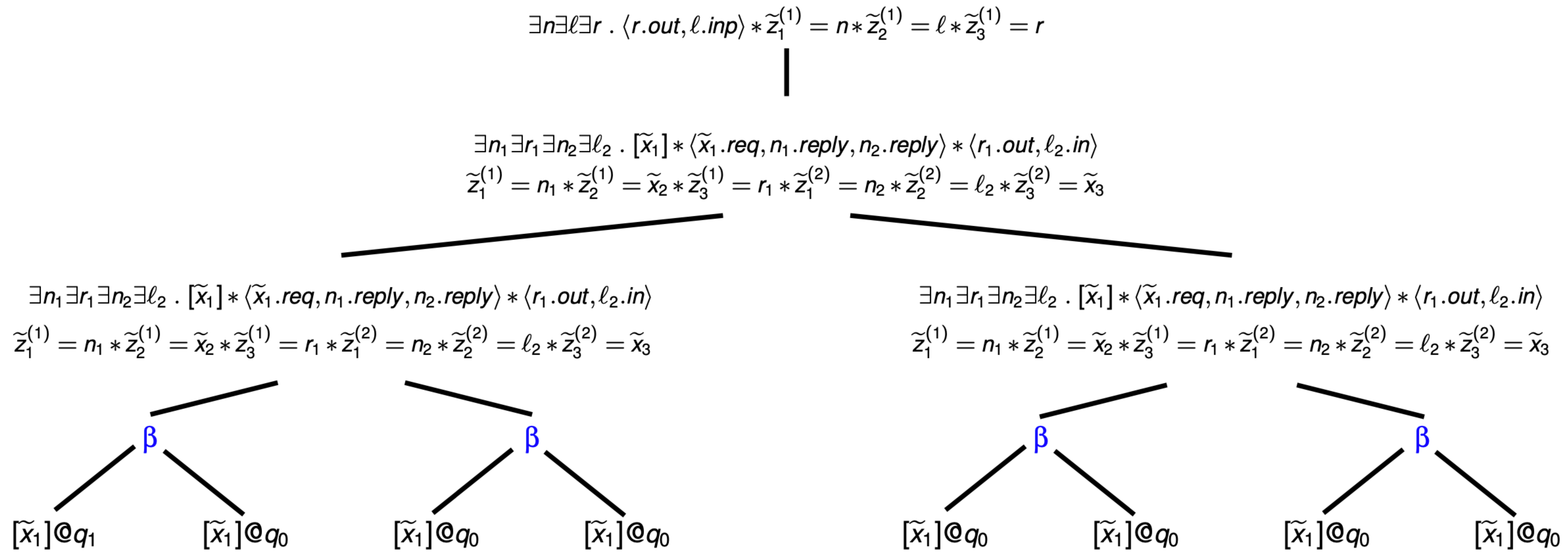
$$(\beta)\quad Node(n,\ell,r) \leftarrow \exists n_1 \exists r_1 \exists n_2 \exists \ell_2\, .\, [n] * \langle n.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle * Node(n_1,\ell,r_1) * Node(n_2,\ell_2,r)$$

$$(\gamma_0)\quad Node(n,\ell,r) \leftarrow [n]@q_0 * n = \ell * n = r \qquad (\gamma_1)\quad Node(n,\ell,r) \leftarrow [n]@q_1 * n = \ell * n = r$$



A Tree with Leaves Linked in a Ring

$$
\begin{array}{lll}
(\alpha) & Root() & \leftarrow \exists n \exists \ell \exists r \,.\, \langle r.out, \ell.in \rangle * Node(n,\ell,r) \\
(\beta) & Node(n,\ell,r) & \leftarrow \exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [n] * \langle n.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle * Node(n_1,\ell,r_1) * Node(n_2,\ell_2,r) \\
(\gamma_0) & Node(n,\ell,r) & \leftarrow [n]@q_0 * n = \ell * n = r \qquad (\gamma_1) \quad Node(n,\ell,r) \leftarrow [n]@q_1 * n = \ell * n = r
\end{array}
$$



$\exists n \exists \ell \exists r \,.\, \langle r.out, \ell.inp \rangle * \tilde{z}_1^{(1)} = n * \tilde{z}_2^{(1)} = \ell * \tilde{z}_3^{(1)} = r$

$\exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [\tilde{x}_1] * \langle \tilde{x}_1.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle$
$\tilde{z}_1^{(1)} = n_1 * \tilde{z}_2^{(1)} = \tilde{x}_2 * \tilde{z}_3^{(1)} = r_1 * \tilde{z}_1^{(2)} = n_2 * \tilde{z}_2^{(2)} = \ell_2 * \tilde{z}_3^{(2)} = \tilde{x}_3$

$\exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [\tilde{x}_1] * \langle \tilde{x}_1.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle$
$\tilde{z}_1^{(1)} = n_1 * \tilde{z}_2^{(1)} = \tilde{x}_2 * \tilde{z}_3^{(1)} = r_1 * \tilde{z}_1^{(2)} = n_2 * \tilde{z}_2^{(2)} = \ell_2 * \tilde{z}_3^{(2)} = \tilde{x}_3$

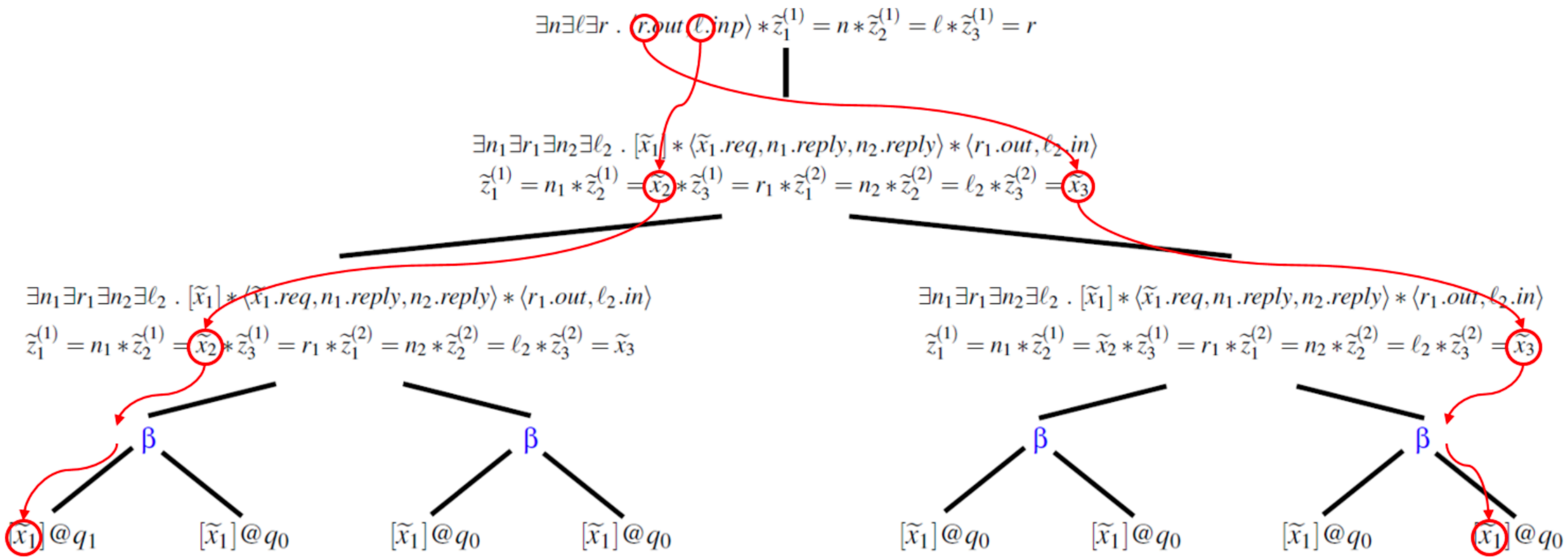
$\exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [\tilde{x}_1] * \langle \tilde{x}_1.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle$
$\tilde{z}_1^{(1)} = n_1 * \tilde{z}_2^{(1)} = \tilde{x}_2 * \tilde{z}_3^{(1)} = r_1 * \tilde{z}_1^{(2)} = n_2 * \tilde{z}_2^{(2)} = \ell_2 * \tilde{z}_3^{(2)} = \tilde{x}_3$

β β β β

$[\tilde{x}_1]@q_1$ $[\tilde{x}_1]@q_0$ $[\tilde{x}_1]@q_0$ $[\tilde{x}_1]@q_0$ $[\tilde{x}_1]@q_0$ $[\tilde{x}_1]@q_0$ $[\tilde{x}_1]@q_0$ $[\tilde{x}_1]@q_0$

Havoc Action as Tree Transductions

- Non-deterministically choses which interaction <$x_1.p_1$ ... $x_n.p_n$> is triggered
- Tracks each variable x_i to the atom [x]@q that instantiates it (creates the respective node)
- Change the states of these nodes according to the transitions of the behavior (state machine)



$\exists n \exists \ell \exists r \,.\, \langle r.out, \ell.inp \rangle * \widetilde{z}_1^{(1)} = n * \widetilde{z}_2^{(1)} = \ell * \widetilde{z}_3^{(1)} = r$
$\exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [\widetilde{x}_1] * \langle \widetilde{x}_1.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle$
$\widetilde{z}_1^{(1)} = n_1 * \widetilde{z}_2^{(1)} = \widetilde{x}_2 * \widetilde{z}_3^{(1)} = r_1 * \widetilde{z}_1^{(2)} = n_2 * \widetilde{z}_2^{(2)} = \ell_2 * \widetilde{z}_3^{(2)} = \widetilde{x}_3$
$\exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [\widetilde{x}_1] * \langle \widetilde{x}_1.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle$
$\widetilde{z}_1^{(1)} = n_1 * \widetilde{z}_2^{(1)} = \widetilde{x}_2 * \widetilde{z}_3^{(1)} = r_1 * \widetilde{z}_1^{(2)} = n_2 * \widetilde{z}_2^{(2)} = \ell_2 * \widetilde{z}_3^{(2)} = \widetilde{x}_3$
$\exists n_1 \exists r_1 \exists n_2 \exists \ell_2 \,.\, [\widetilde{x}_1] * \langle \widetilde{x}_1.req, n_1.reply, n_2.reply \rangle * \langle r_1.out, \ell_2.in \rangle$
$\widetilde{z}_1^{(1)} = n_1 * \widetilde{z}_2^{(1)} = \widetilde{x}_2 * \widetilde{z}_3^{(1)} = r_1 * \widetilde{z}_1^{(2)} = n_2 * \widetilde{z}_2^{(2)} = \ell_2 * \widetilde{z}_3^{(2)} = \widetilde{x}_3$
β
β
β
β
$[\widetilde{x}_1]@q_1$
$[\widetilde{x}_1]@q_0$
$[\widetilde{x}_1]@q_0$
$[\widetilde{x}_1]@q_0$
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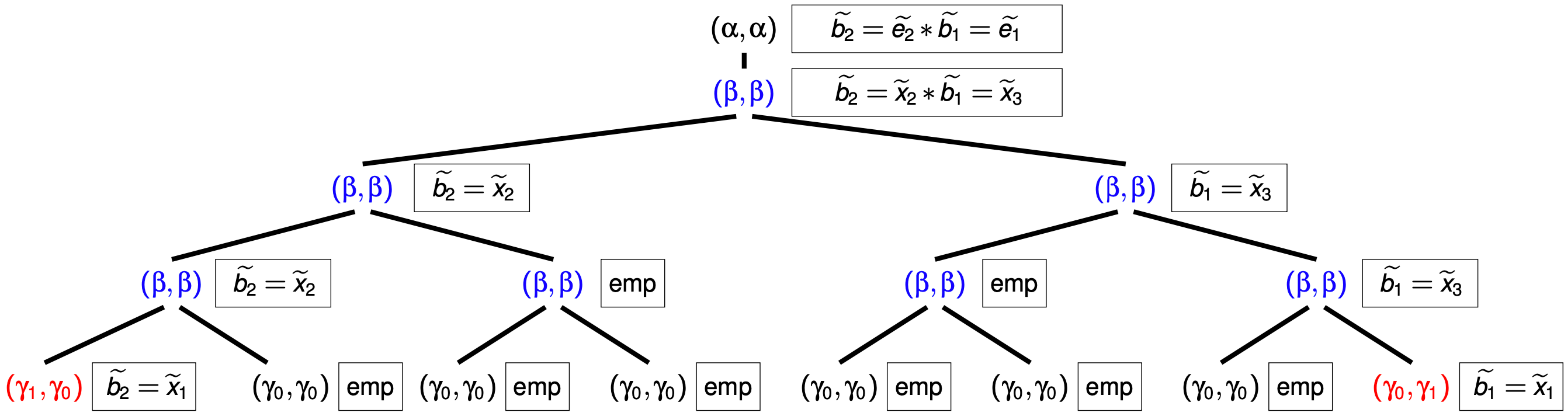
$[\widetilde{x}_1]@q_0$
$[\widetilde{x}_1]@q_0$
$[\widetilde{x}_1]@q_0$

Havoc Action as Tree Transductions

- Non-deterministically choses which interaction <$x_1.p_1$ … $x_n.p_n$> is triggered
- Tracks each variable x_i to the atom [x]@q that instantiates it (creates the respective node)
- Change the states of these nodes according to the transitions of the behavior (state machine)


(α,α)
$\widetilde{b}_2 = \widetilde{e}_2 * \widetilde{b}_1 = \widetilde{e}_1$
(β,β)
$\widetilde{b}_2 = \widetilde{x}_2 * \widetilde{b}_1 = \widetilde{x}_3$
(β,β)
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emp
(γ_0,γ_0)
emp
(γ_0,γ_1)
$\widetilde{b}_1 = \widetilde{x}_1$

End of Part I

A simplified model of dynamic reconfigurable systems

- components with finite-state behavior and interactions of finite arity
- a sequential programming language for describing reconfiguration

A resource logic for describing possibly infinite sets of configurations

- inductively defined predicates

A proof system for reconfiguration programs

- uses local reasoning to a maximum extent
- generates external proof obligations (entailments)

Entailment Checking Between Inductive Sets of Configurations

Key to mechanising proof generation for reconfiguration programs

- checking havoc invariance requires entailment checking
- entailments is needed when applying the standard consequence rule of Hoare logic
- solving frame inference (conditional rule) uses similar techniques

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Entailment of inductively defined predicates is a hard problem [Bozga, Bueri, I IJCAR'22]

- satisfiability is decidable (2EXP∩NP-hard)
- entailment is undecidable in general and decidable under certain restrictions (4EXP∩2EXP-hard)
- we currently try to understand what are the weakest such restrictions

Relational Structures

$\sum = \{\underbrace{R_1, \dots, R_N}_{\text{relation symbols}}, \underbrace{c_1, \dots, c_M}_{\text{constants}}\}$ relational signature

$S = (\underbrace{U}_{\text{universe}}, \underbrace{\sigma}_{\text{interpretation of symbols from } \sum})$ structure

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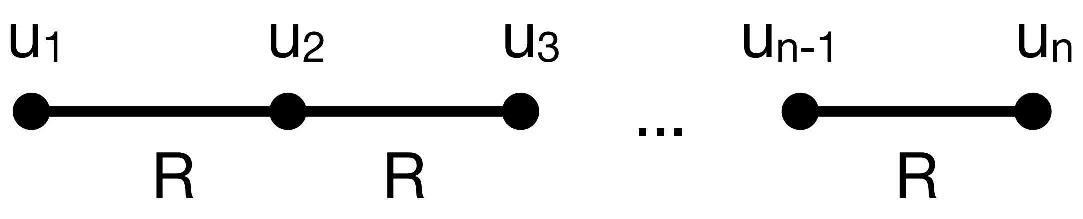
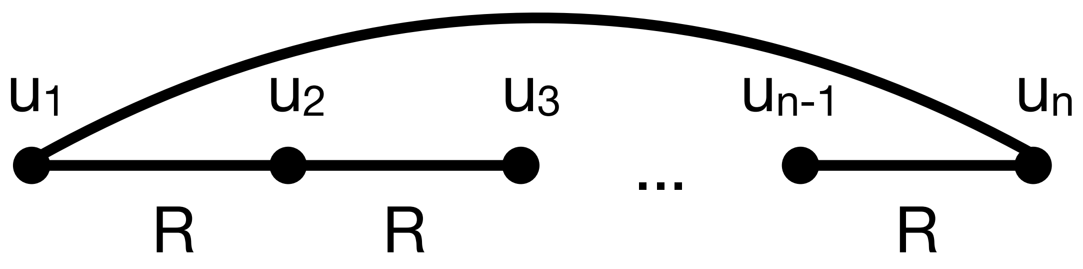
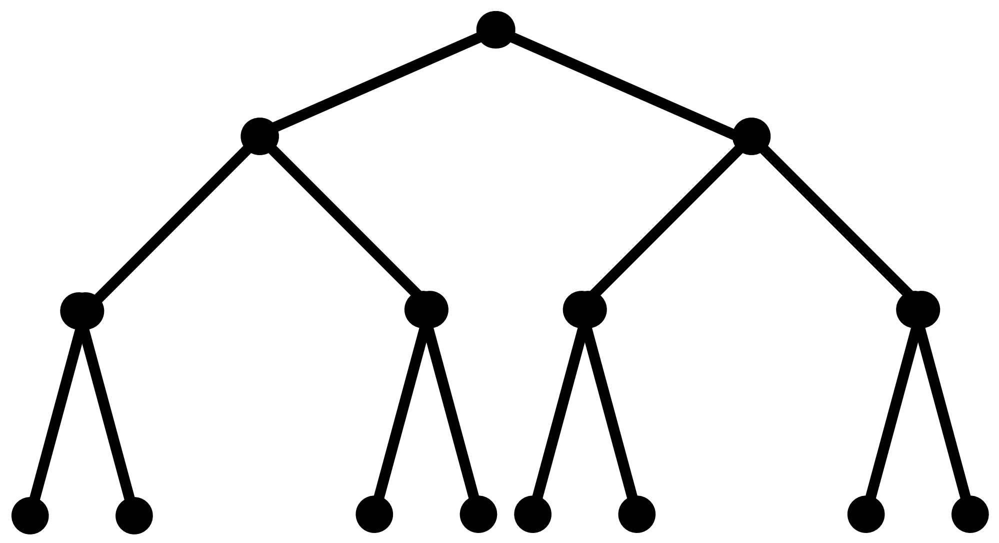


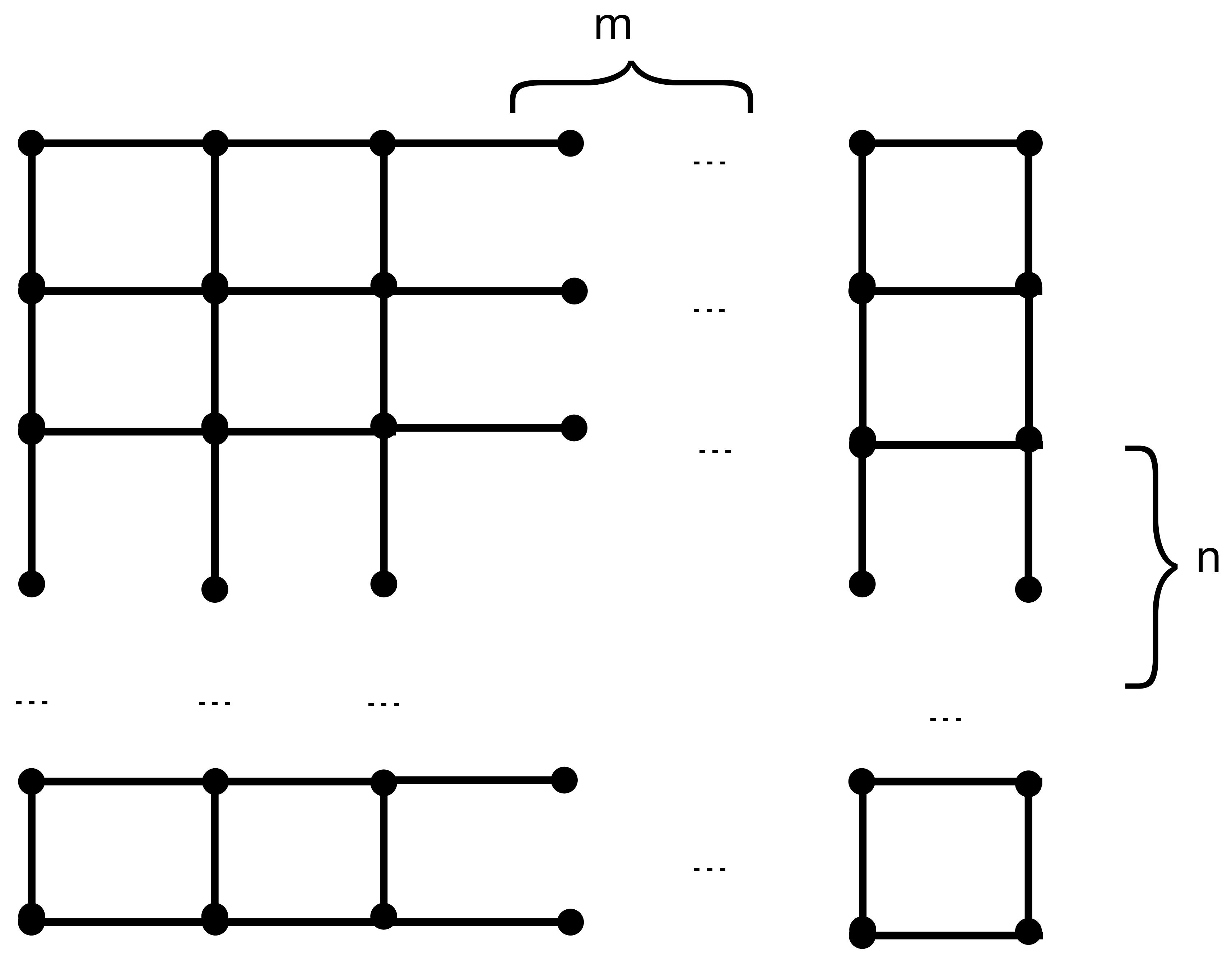
tree-width = 1



tree-width = 2



tree-width = 1



tree-width = min(n, m)

Separation Logic of Relations (SLR)

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$\phi_1 * \phi_2$ — any structure $S_1 \otimes S_2$, such that $S_i \models \phi_i$, for all $i=1,2$

- $(U_1,\sigma_1) \otimes (U_2,\sigma_2) = (U_1 \cup U_2, \sigma_1 \uplus \sigma_2)$
- $\sigma_1 \uplus \sigma_2$ is the point-wise disjoint union of interpretations

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$R_1(y_1, \ldots, y_n) * R_1(z_1, \ldots, z_n)$ implies $y_i \neq z_i$, for at least one $i=1, \ldots, n$

(Monadic) Second Order Logic

$\sum = \{\underbrace{R_1, \dots, R_N}_{\text{relation symbols}}, \underbrace{c_1, \dots., c_M}_{\text{constants}}\}$ relational signature

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$R(x_1, \dots, x_n)$	R contains the tuple of values $x_1, \dots, x_n$, ‣ the rest of the structure remains unspecified
$\exists x.\phi(x)$	quantification over individual elements of U
$\exists X.\phi(X)$	quantification over relations, i.e., subsets of $\underbrace{\mathrm{U\ x\ \dots\ x\ U}}_{\#(X)}$
$\neg\phi,\ \phi_1 \wedge \phi_2$	boolean connectives

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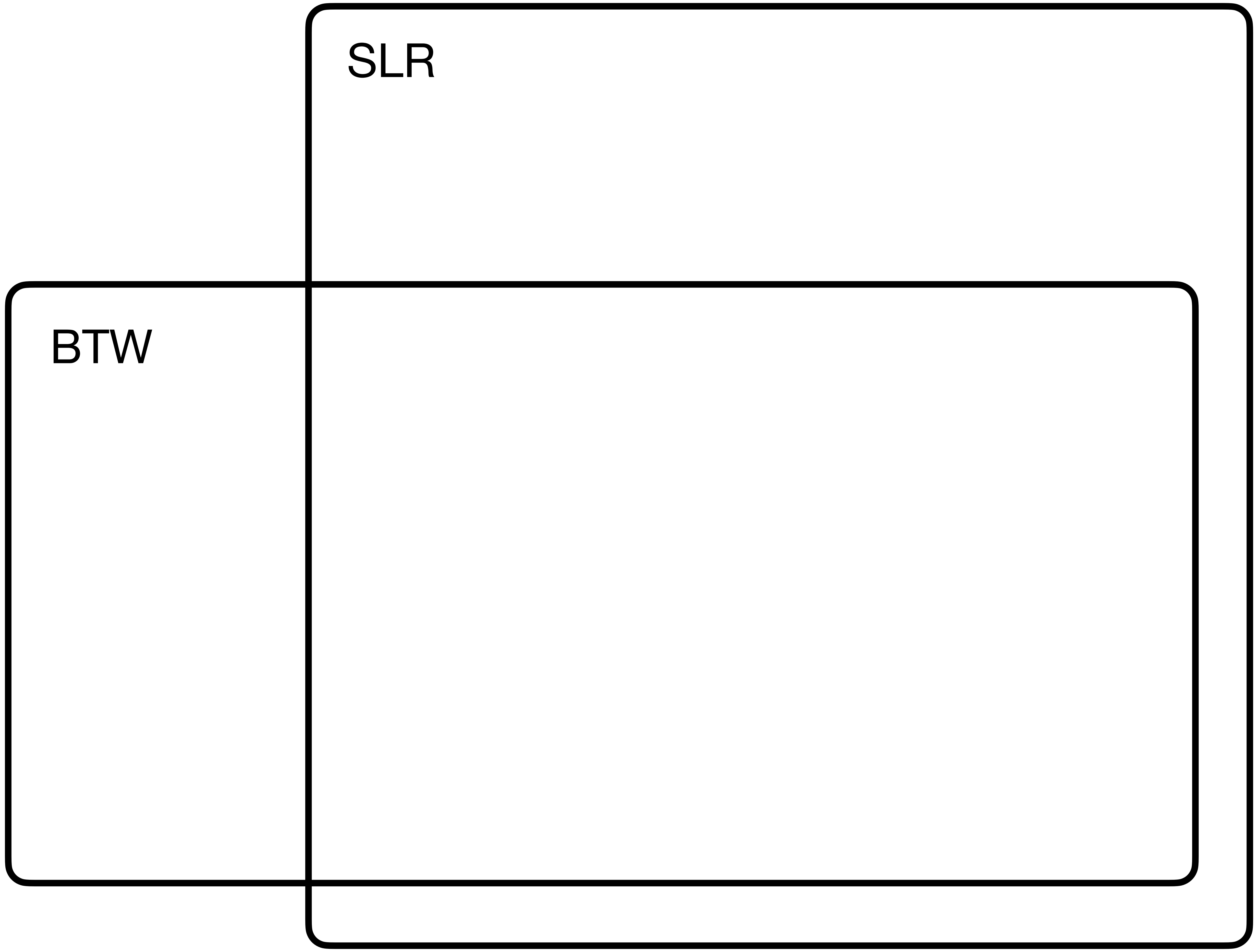
MSO is the yardstick of graph description logics:

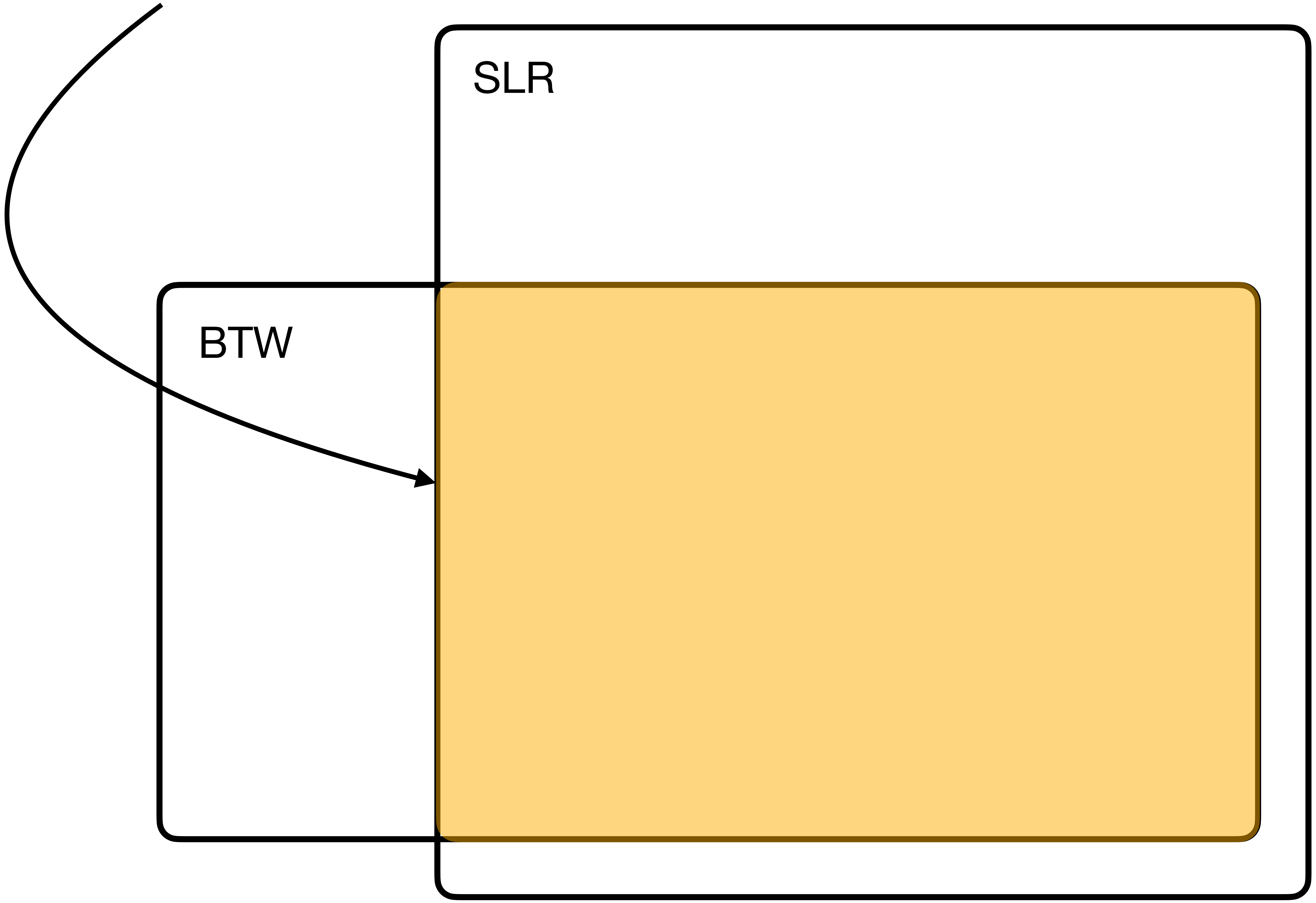
- Decidable for structures of bounded tree-width [Courcelle'90]
- Each class of structures with a decidable MSO theory has bounded tree-width [Seese'91]

The Big Picture



The Big Picture

A decidable characterization
[Bozga, Bueri, I, Zuleger ARXIV 2023a]



Canonical Models

ls(x,y) ← ∃z . R(x,z) * ls(z,y)

ls(x,y) ← emp * x=y

Canonical Models

$ls(x,y) \leftarrow \exists z \,.\, R(x,z) * ls(z,y)$

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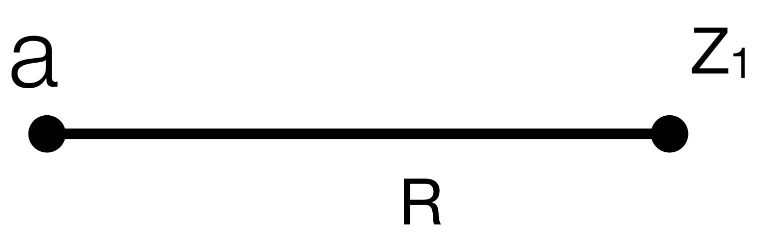
$ls(a,b) \Rightarrow \exists z_1 \,.\, R(a,z_1) * ls(z_1,b)$



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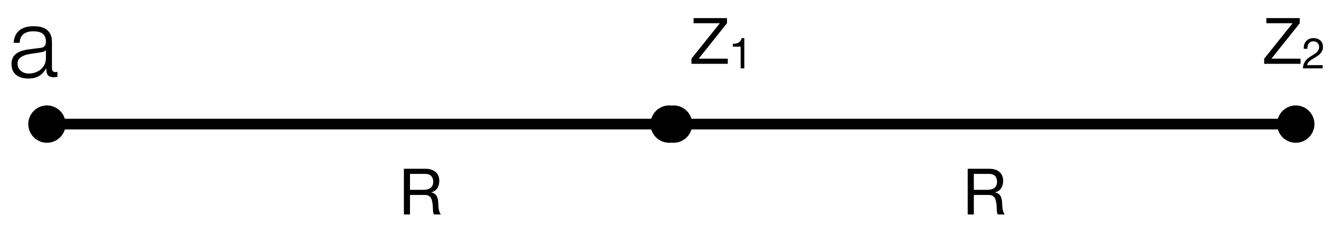

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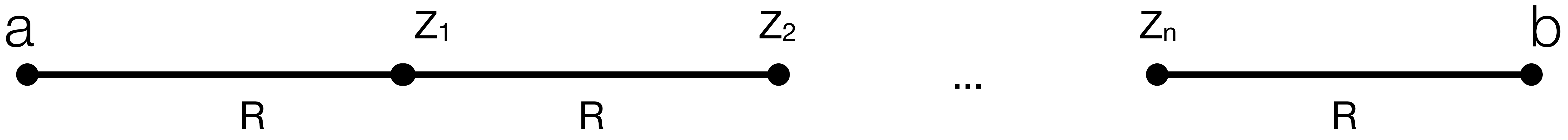

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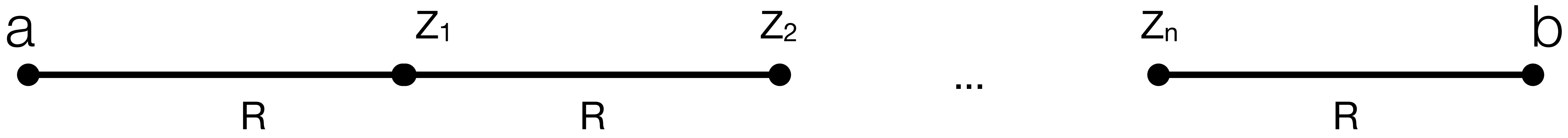


Existentially quantified variables introduced by the unfolding are instantiated by distinct elements

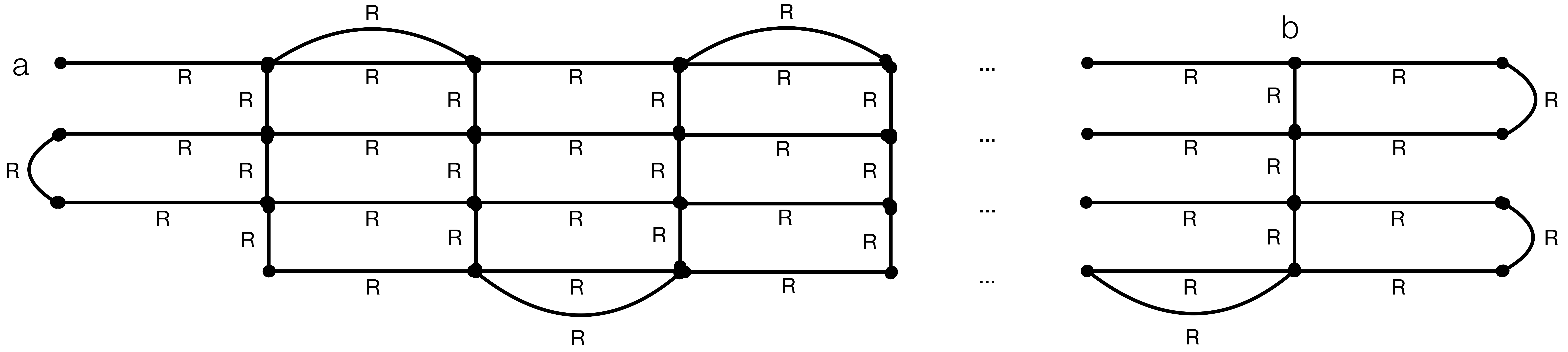
- there exists a uniform bound on the tree-width of canonical models
- the maximal number of variables that occur (free or bound) in an inductive definition

Canonical Models

ls(x,y) ← ∃z . R(x,z) * ls(z,y)

ls(x,y) ← emp * x=y

ls(a,b) ⇒ ∃z$_1$. R(a,z$_1$) * ls(z$_1$,b) ⇒ ∃z$_1$ ∃z$_2$. R(a,z$_1$) * R(z$_1$,z$_2$) * ls(z$_2$,b) ⇒ ... ⇒ ∃z$_1$ ∃z$_2$ ... ∃z$_n$. R(a,z$_1$) * R(z$_1$,z$_2$) * ... * R(z$_n$,b)



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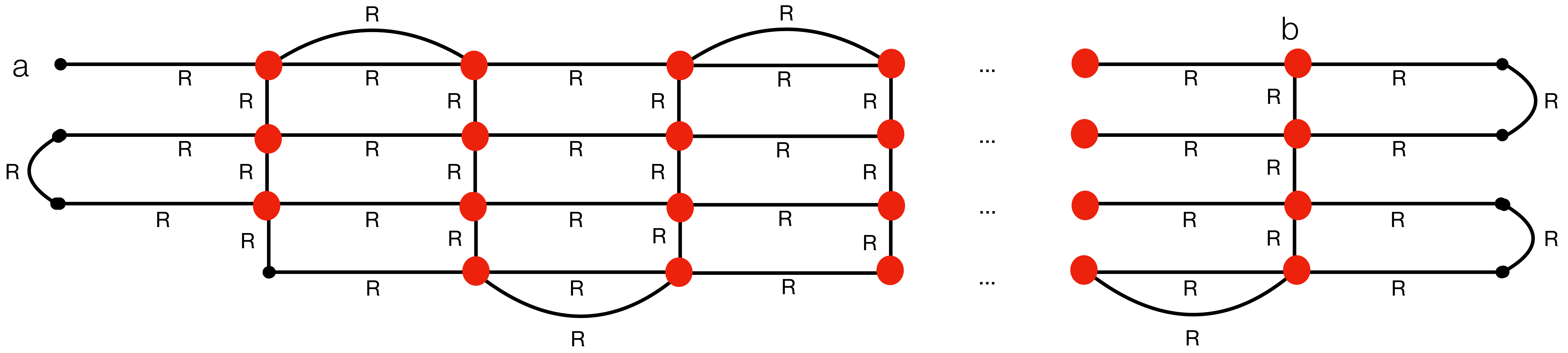
ls(a,b) ⇒ ∃z₁ . R(a,z₁) * ls(z₁,b) ⇒ ∃z₁ ∃z₂ . R(a,z₁) * R(z₁,z₂) * ls(z₂,b) ⇒ … ⇒ ∃z₁ ∃z₂ … ∃zₙ . R(a,z₁) * R(z₁,z₂) * … * R(zₙ,b)



Each model is obtained from a canonical model by internal fusion

- produces unbounded tree-width sets of models

Bounding the Tree-Width

$ls(x,y) \leftarrow \exists z \,.\, D(z) * R(x,z) * ls(z,y)$

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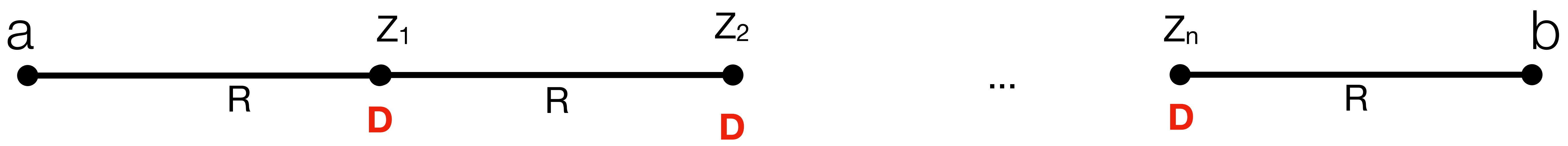

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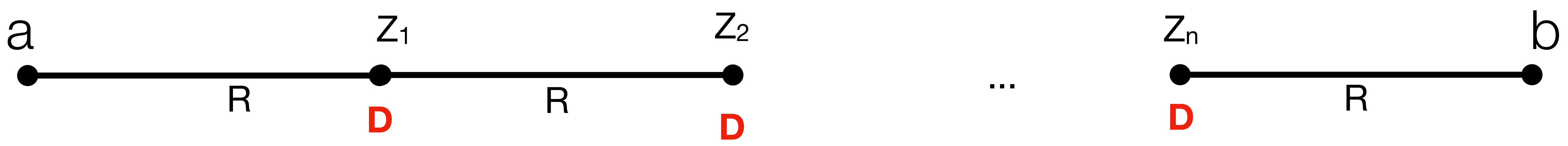

The color of an element = the set of unary relation symbols labeling the element

- only elements with disjoint colors can be fused

Persistent Variables

$\mathrm{ls}(x,y) \leftarrow \exists z \,.\, R(z,y) * R(x,z) * \mathrm{ls}(z,y)$

$\mathrm{ls}(x,y) \leftarrow \mathrm{emp} * x=y$

$\mathrm{ls}(a,b) \Rightarrow \exists z_1 \,.\, R(z_1,b) * R(a,z_1) * \mathrm{ls}(z_1,b)$

$\Rightarrow \exists z_1\, \exists z_2 \,.\, R(z_1,b) * R(a,z_1) * R(z_2,b) * R(z_1,z_2) * \mathrm{ls}(z_2,b)$

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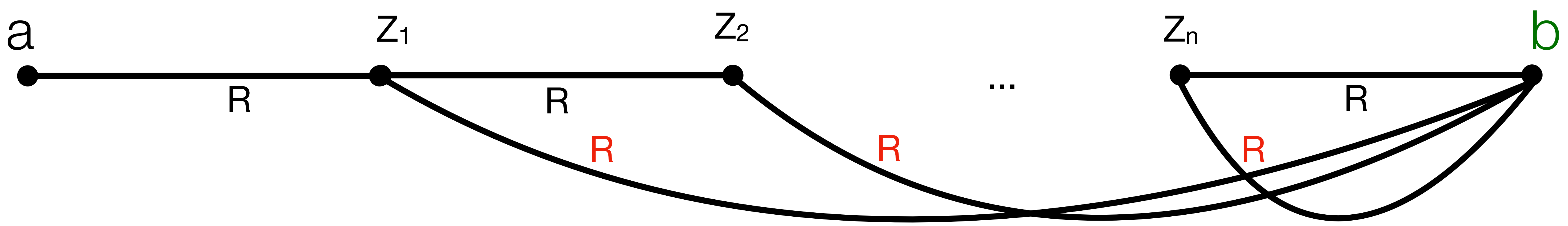
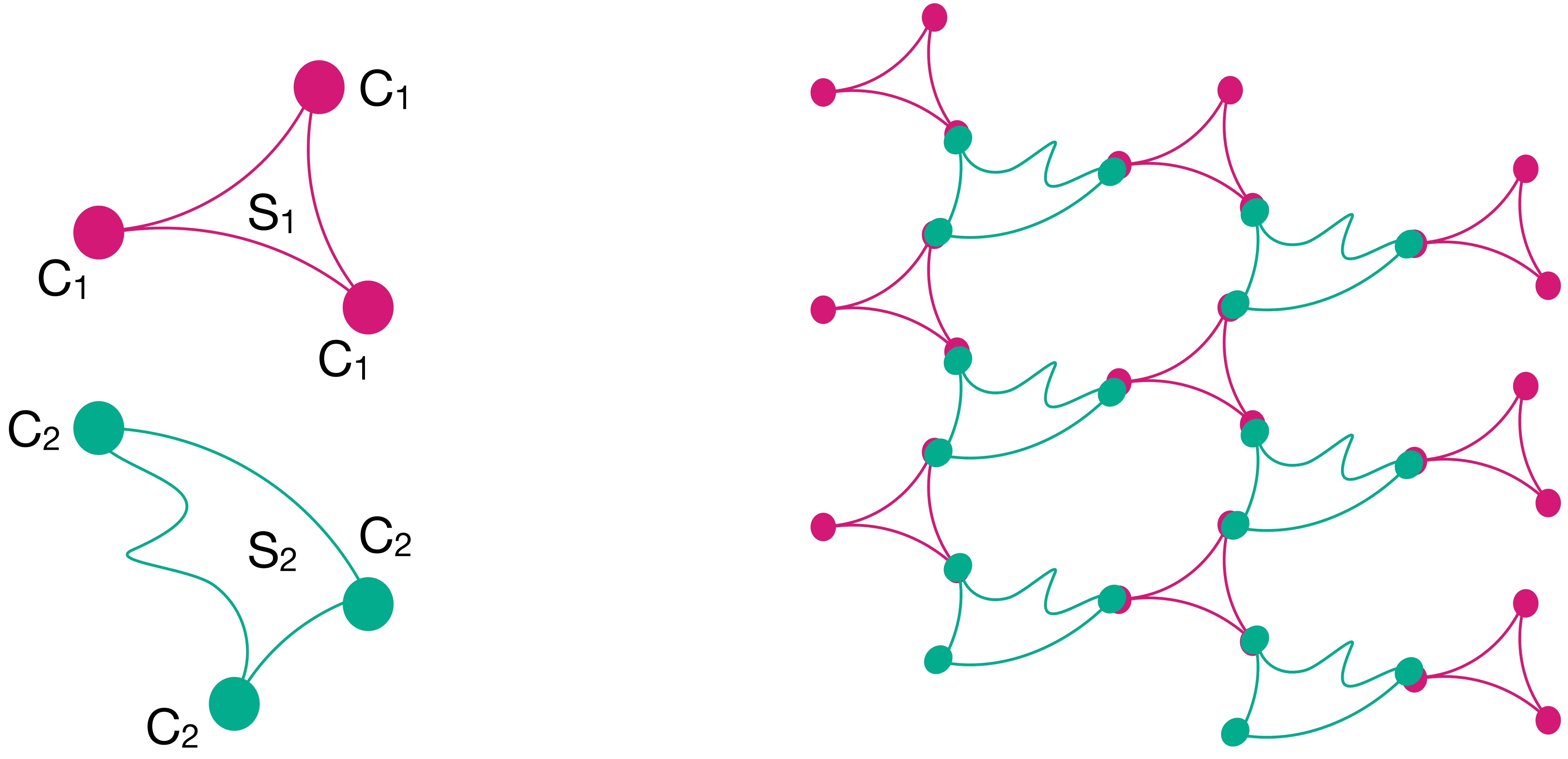
$\Rightarrow \exists z_1\, \exists z_2 \ldots \exists z_n \,.\, R(z_1,b) * R(a,z_1) * R(z_2,b) * R(z_1,z_2) * \ldots * R(z_n,b) * R(z_n,b)$



The color of an element = the set of relation atoms involving only constants besides the element

- persistent variables can be detected by a greatest fixpoint iteration over the set of inductive definitions

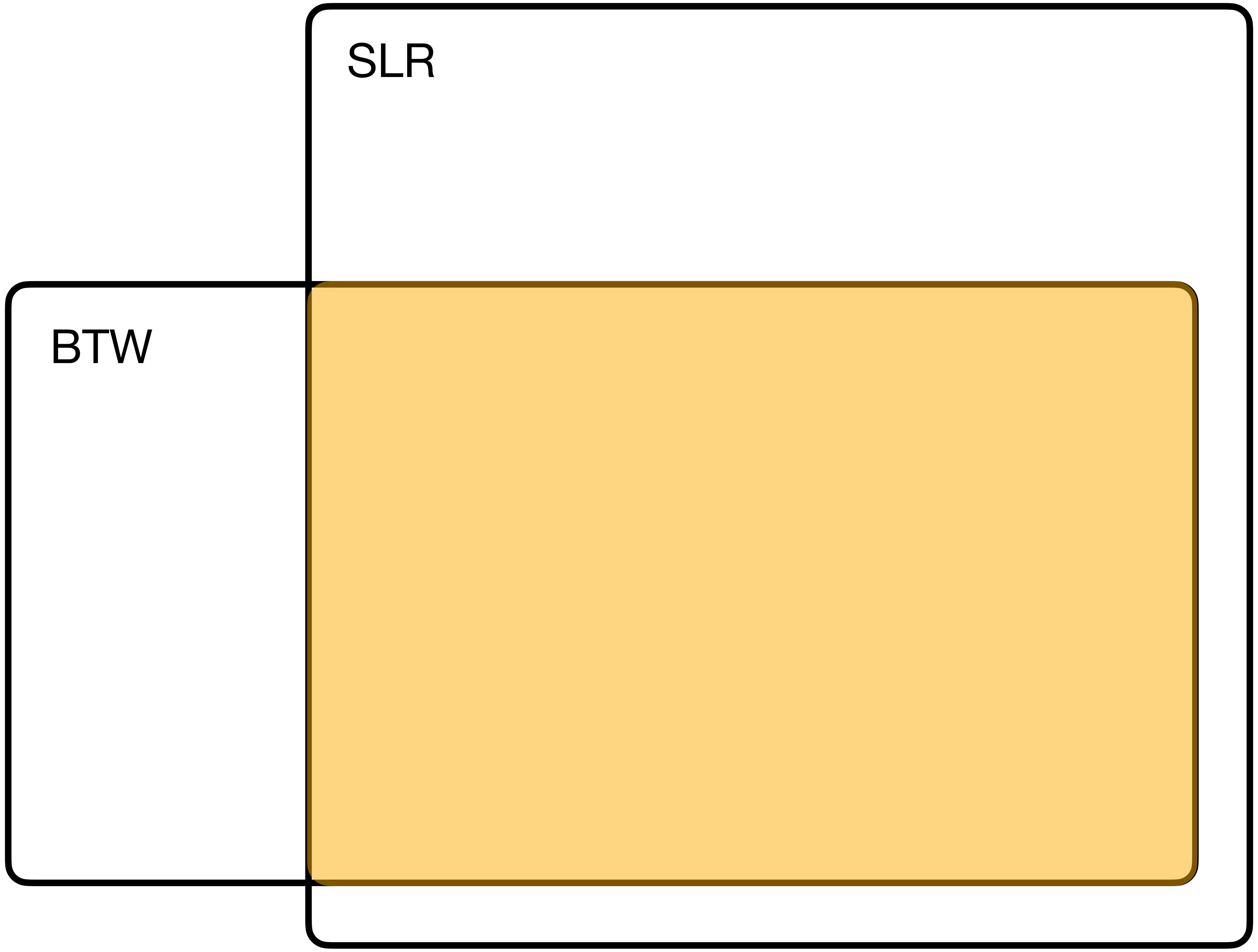
A Decidable Condition



Given an SID Δ, the set of Δ-models of a given sentence ϕ is tree-width unbounded IFF there exist connected structures S_1 and S_2 satisfying the following conditions [Bozga, Bueri, I, Zuleger ARXIV 2023a]:

1. for each $k \geq 1$ there exists $n \geq k$, such that n copies of S_1 and S_2 can be embedded in some Δ-model of ϕ
2. each S_i has at least three occurrences of an element colored C_i, for i = 1,2
3. $C_1 \cap C_2 = \varnothing$

The Big Picture



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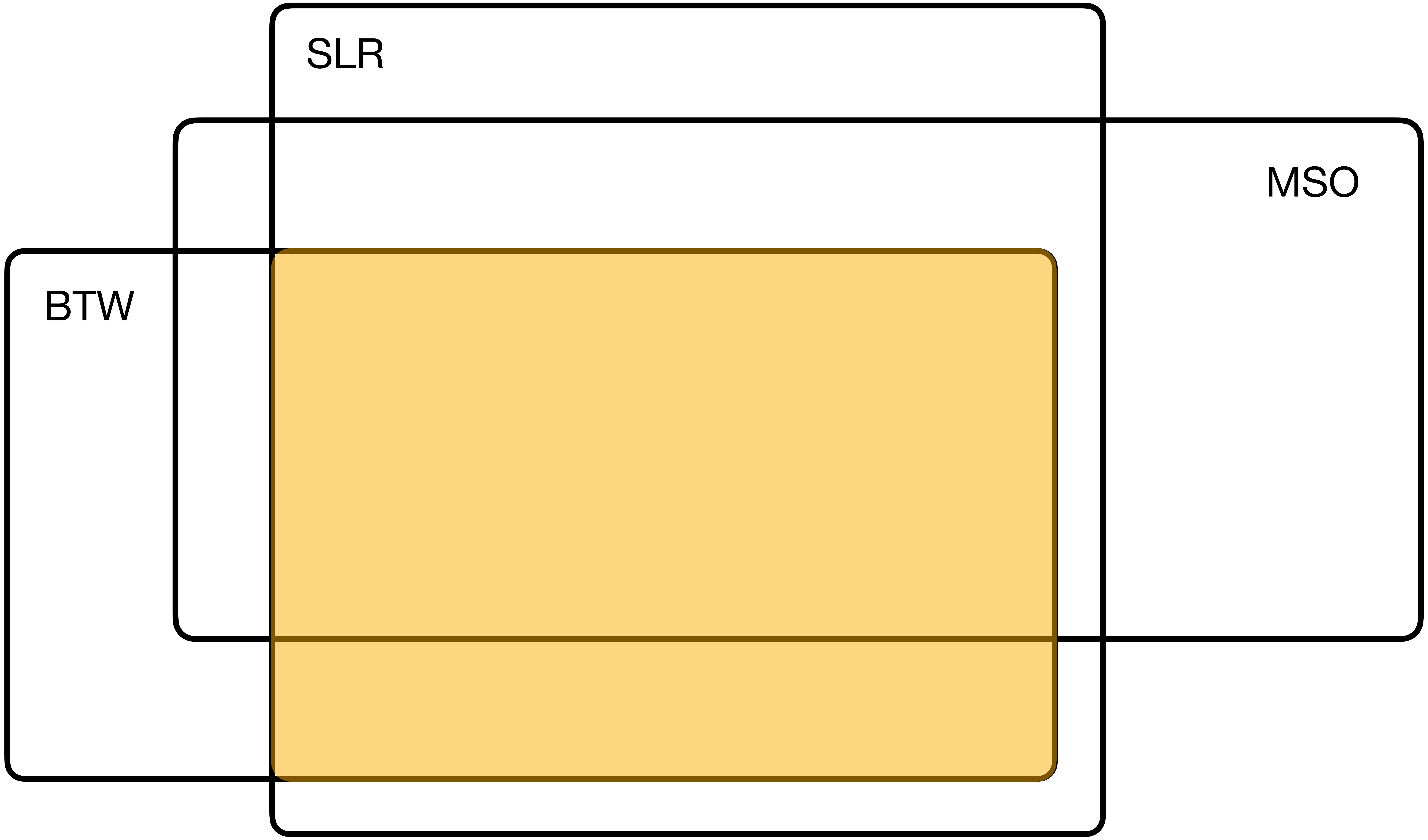

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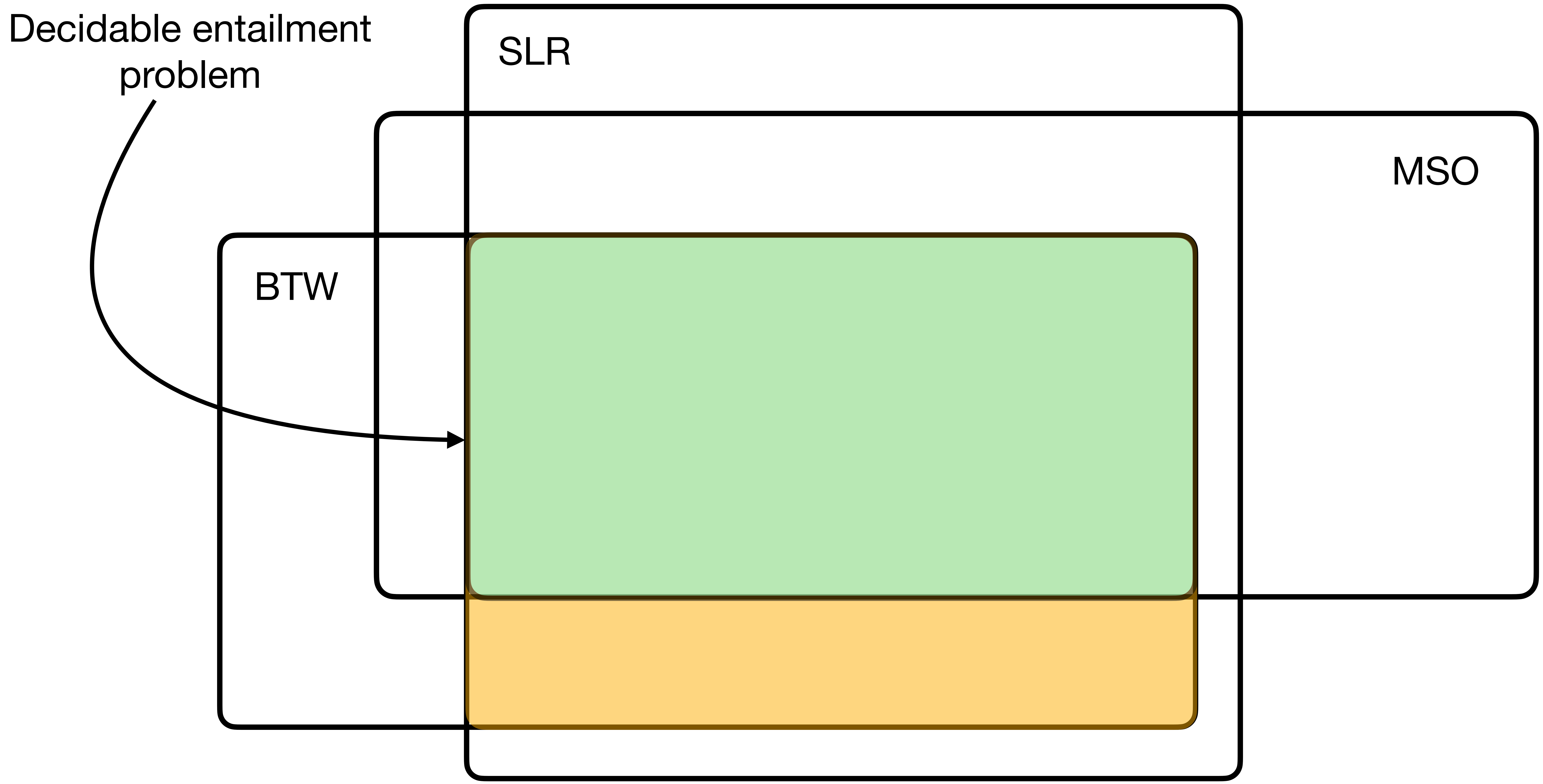


Entailments in MSO ∩ BTW

$$L_1 \overset{?}{\subseteq} L_2$$

Entailments in MSO ∩ BTW

$$\begin{tikzcd} L_1 \arrow[d, dotted] & \overset{?}{\subseteq} & L_2 \arrow[d, dotted] \\ \phi_1 & \overset{?}{\Rightarrow} & \phi_2 \end{tikzcd}$$

Entailments in MSO $\cap$ BTW

$$\begin{tikzcd}
L_1 \arrow[r, phantom, "\overset{?}{\subseteq}"] \arrow[d, dotted] & L_2 \arrow[d, dotted] \\
\phi_1 \arrow[r, phantom, "\overset{?}{\Rightarrow}"] & \phi_2
\end{tikzcd}$$

Is the MSO formula $\phi_1 \wedge \neg\phi_2$ satisfiable ?

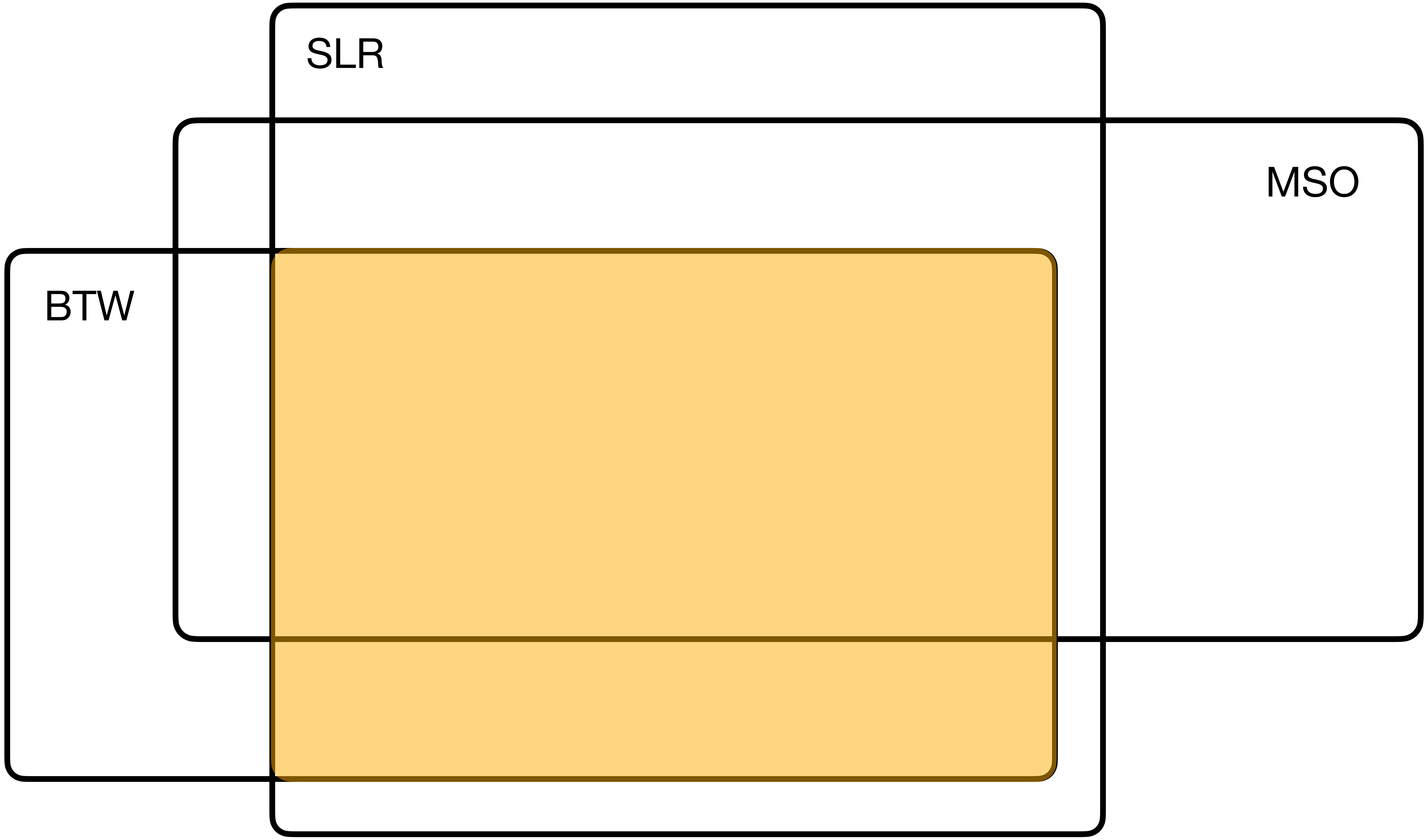
Entailments in MSO ∩ BTW

$$\begin{tikzcd} L_1 \arrow[r, phantom, "\overset{?}{\subseteq}"] \arrow[d, dotted] & L_2 \arrow[d, dotted] \\ \phi_1 \arrow[r, phantom, "\overset{?}{\Rightarrow}"] & \phi_2 \end{tikzcd}$$

Is the MSO formula $\phi_1 \wedge \neg\phi_2$ satisfiable ?

Satisfiability of a MSO formula is decidable over $\{S \mid \mathit{tree\text{-}width}(S) \leqslant k\}$ [Courcelle'90]

The Big Picture



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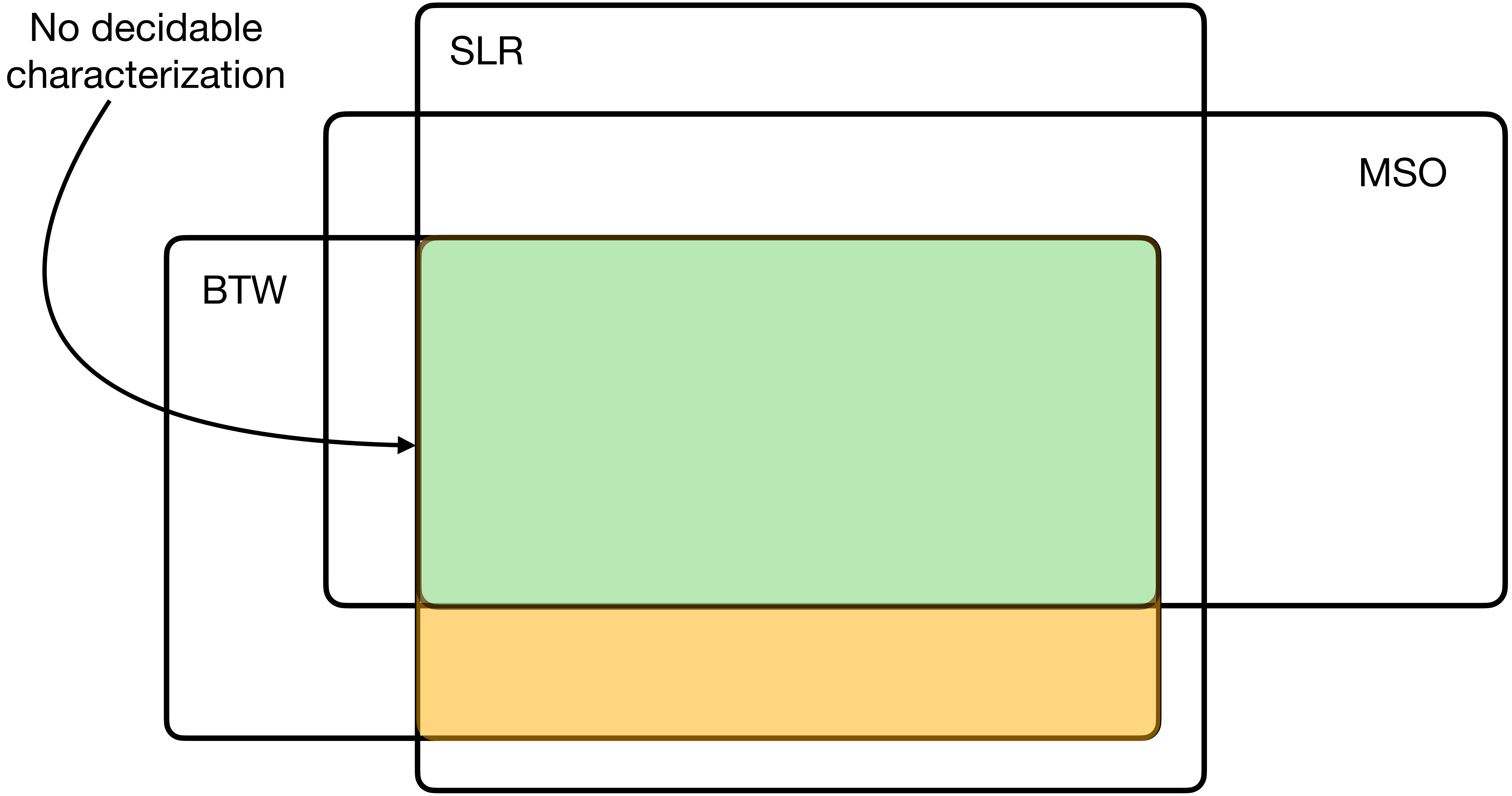
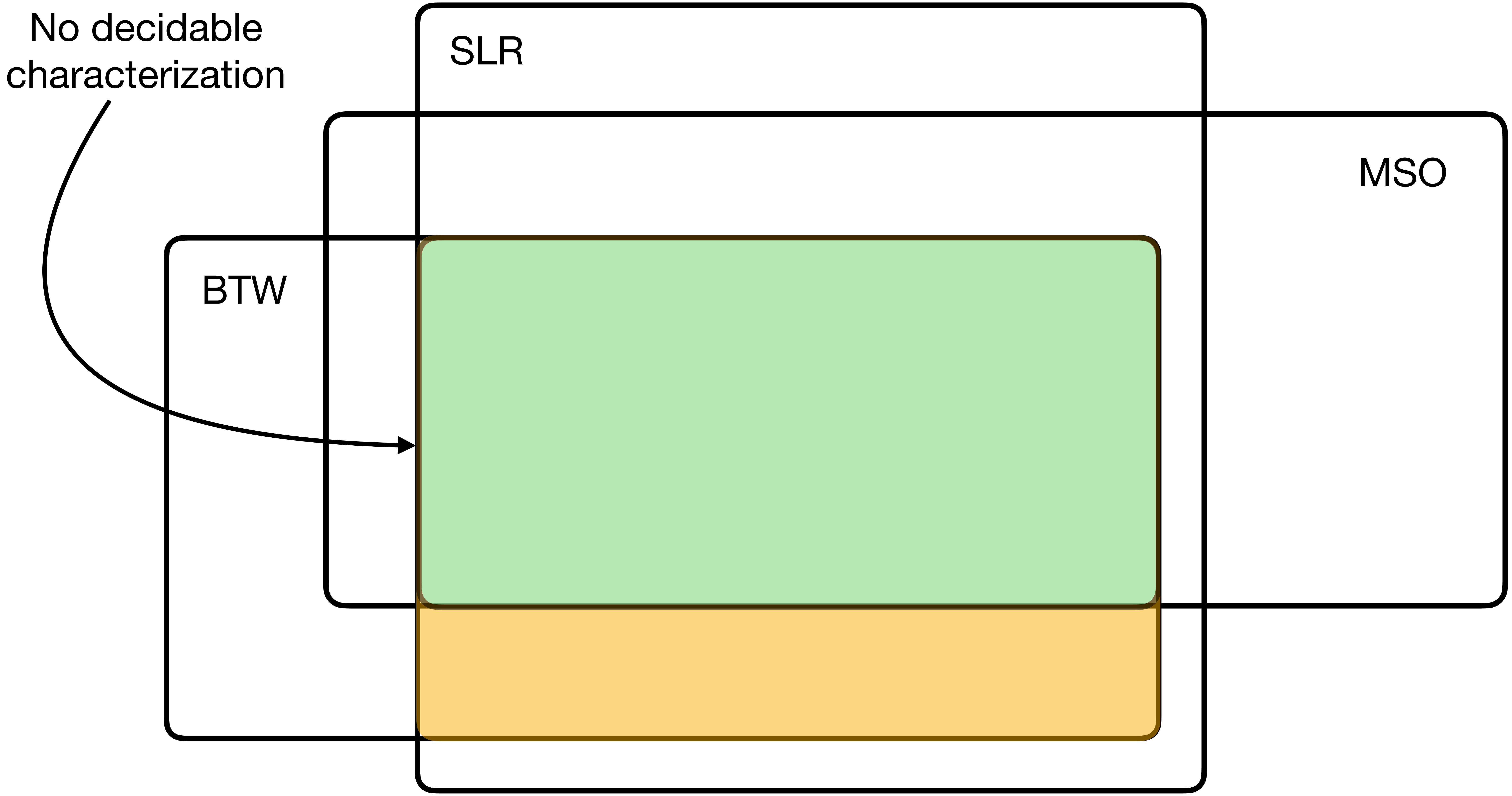


The Big Picture

Given a context-free word language L, the problem “*L is recognizable?”* is undecidable [Greibach’69]



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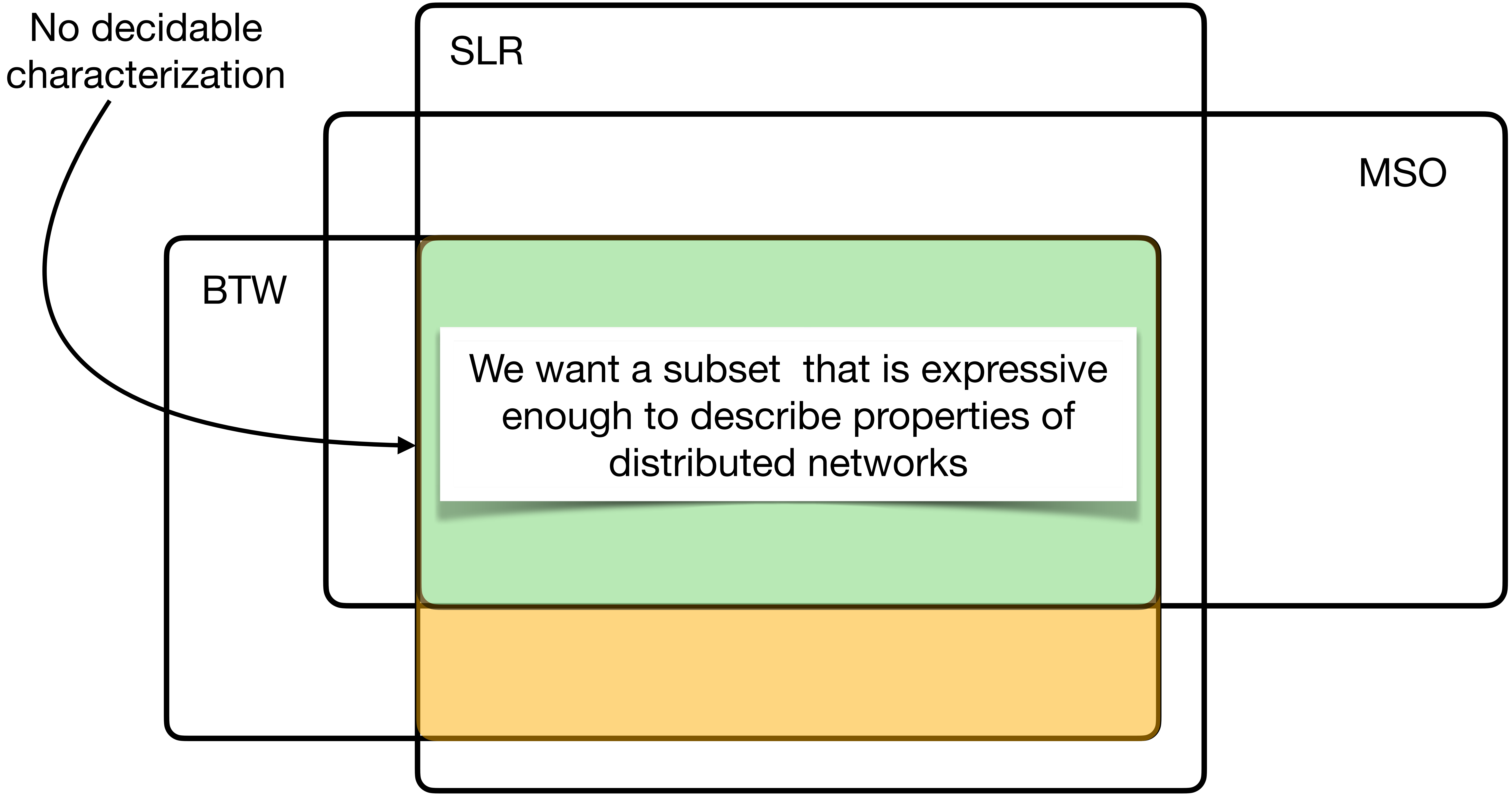
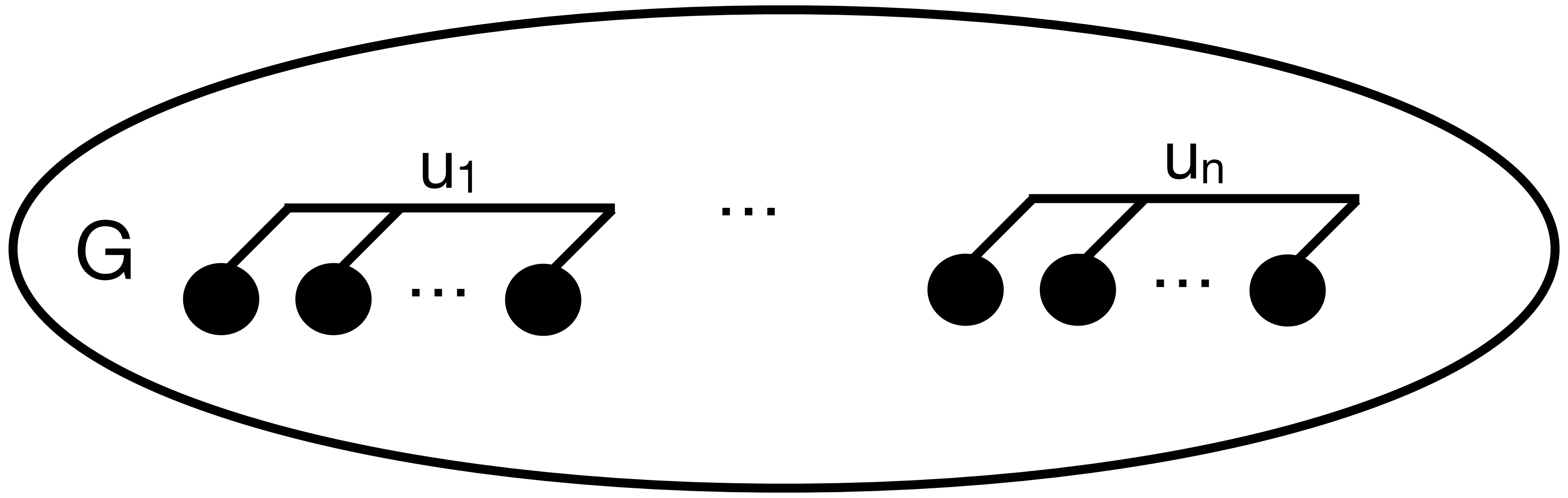


We want a subset that is expressive enough to describe properties of distributed networks

Graph Grammars

Hyperedge-replacement (HR) grammars with operations of the form $(G,u_1,\ldots,u_n)$ and $||_k$



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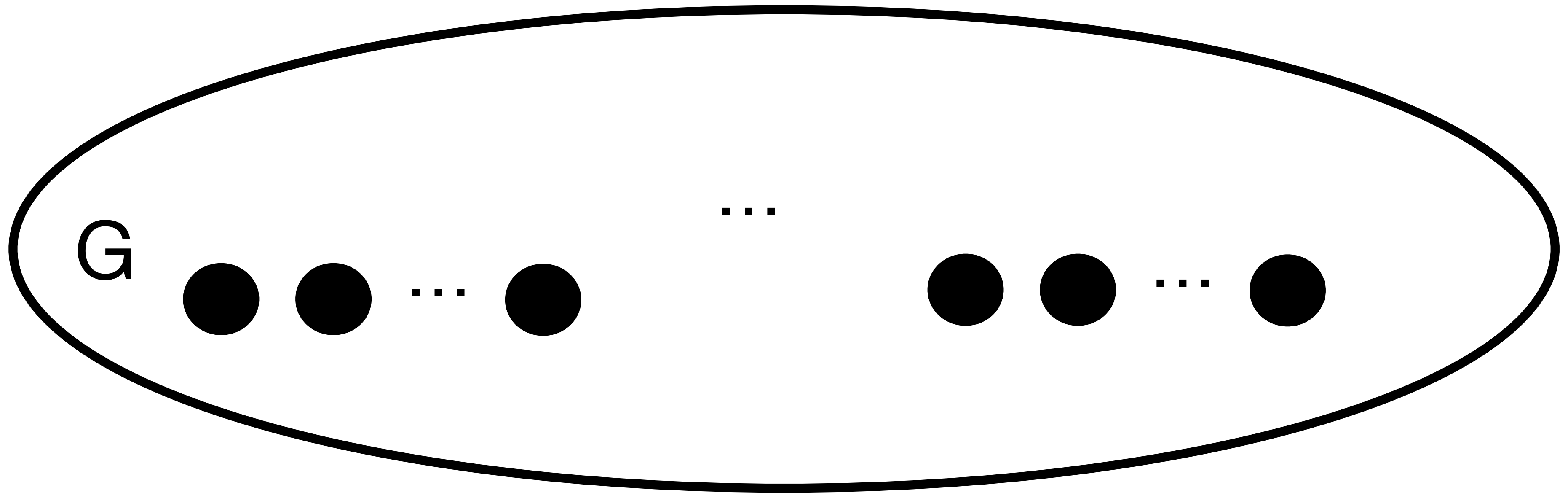

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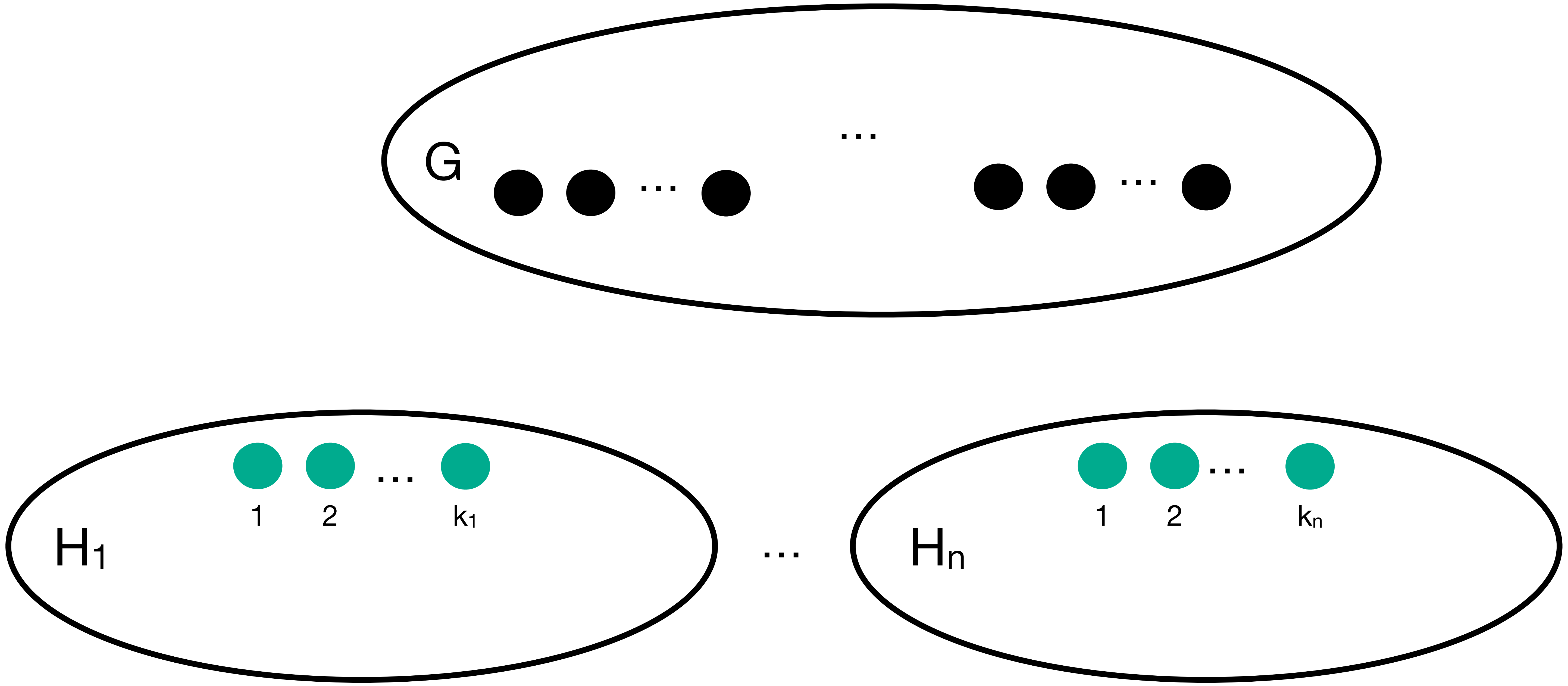

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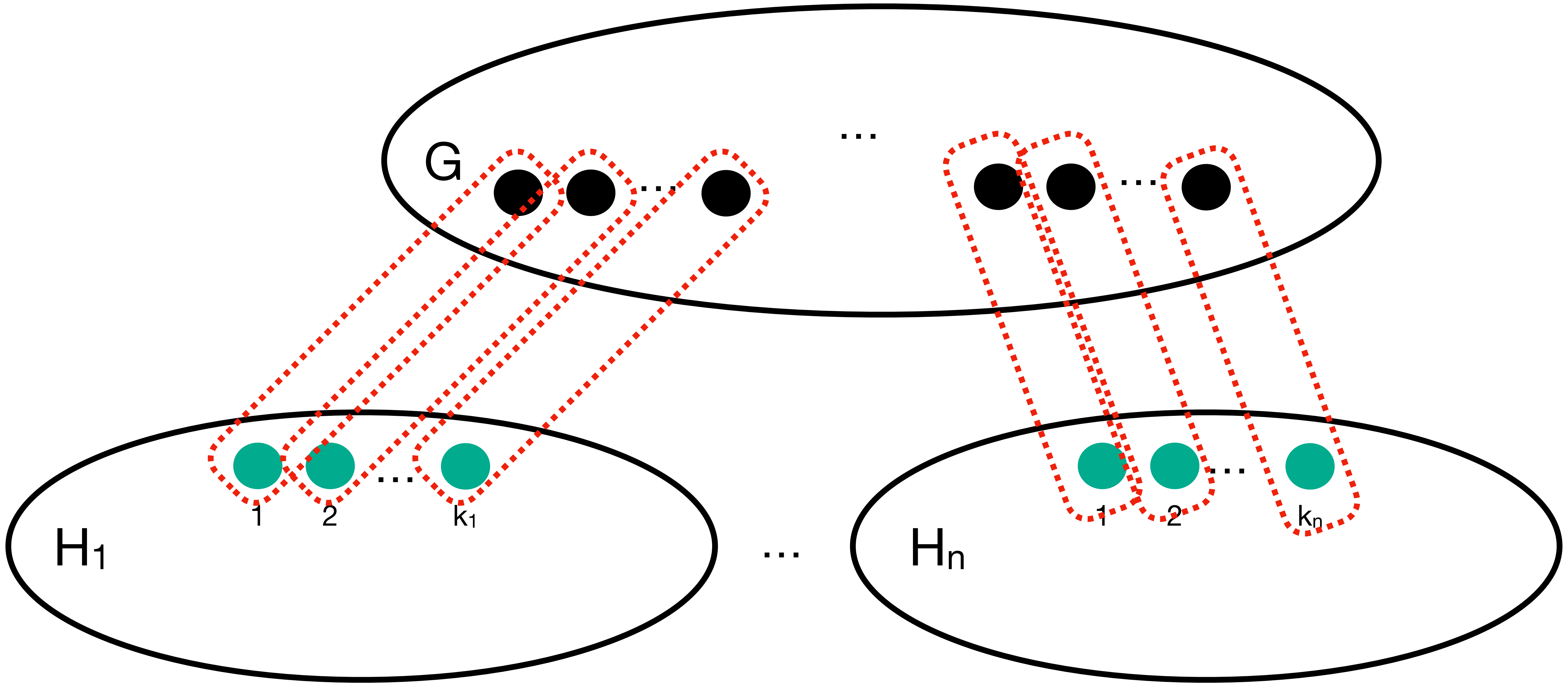

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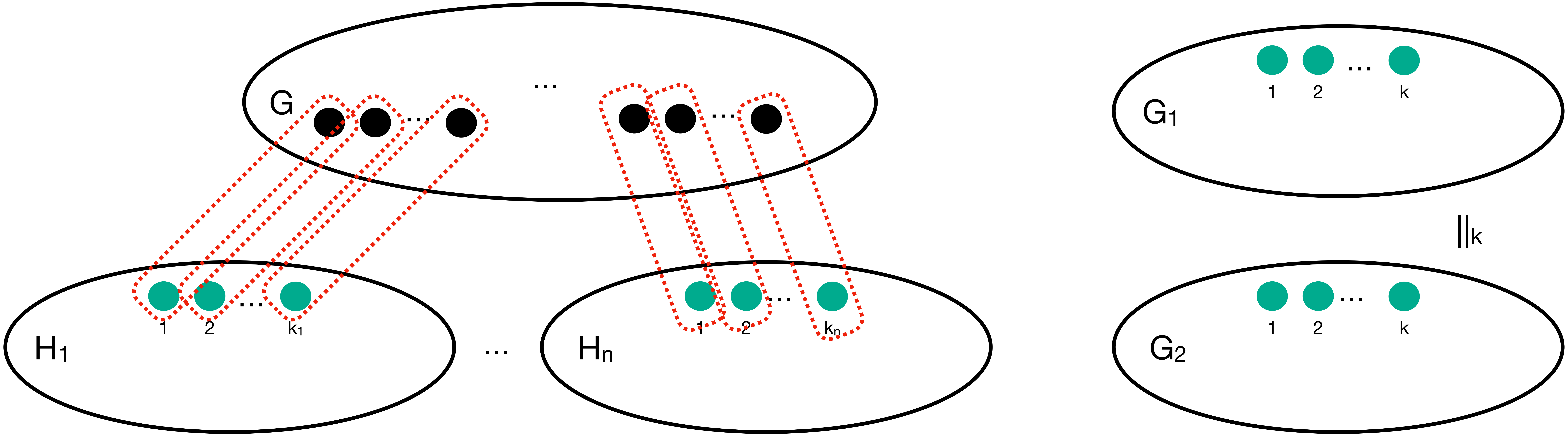

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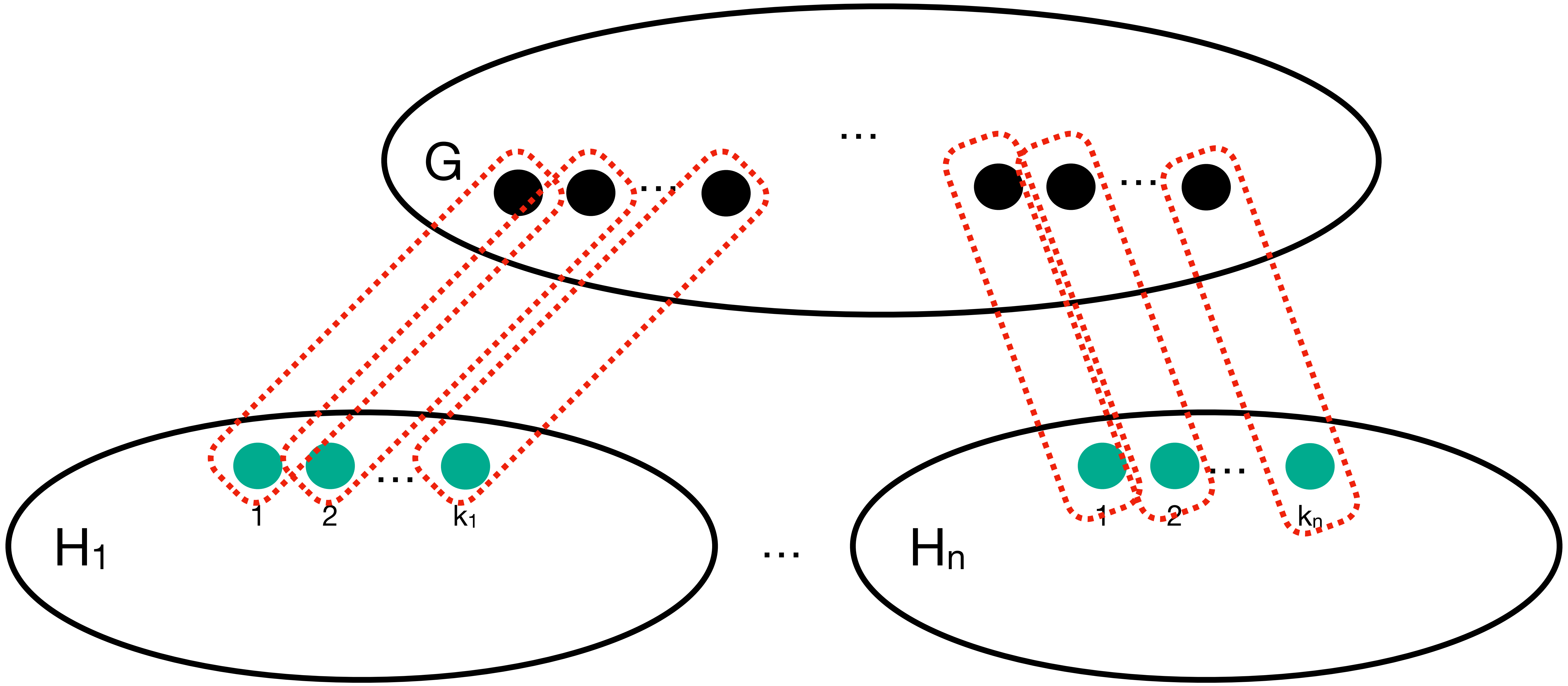

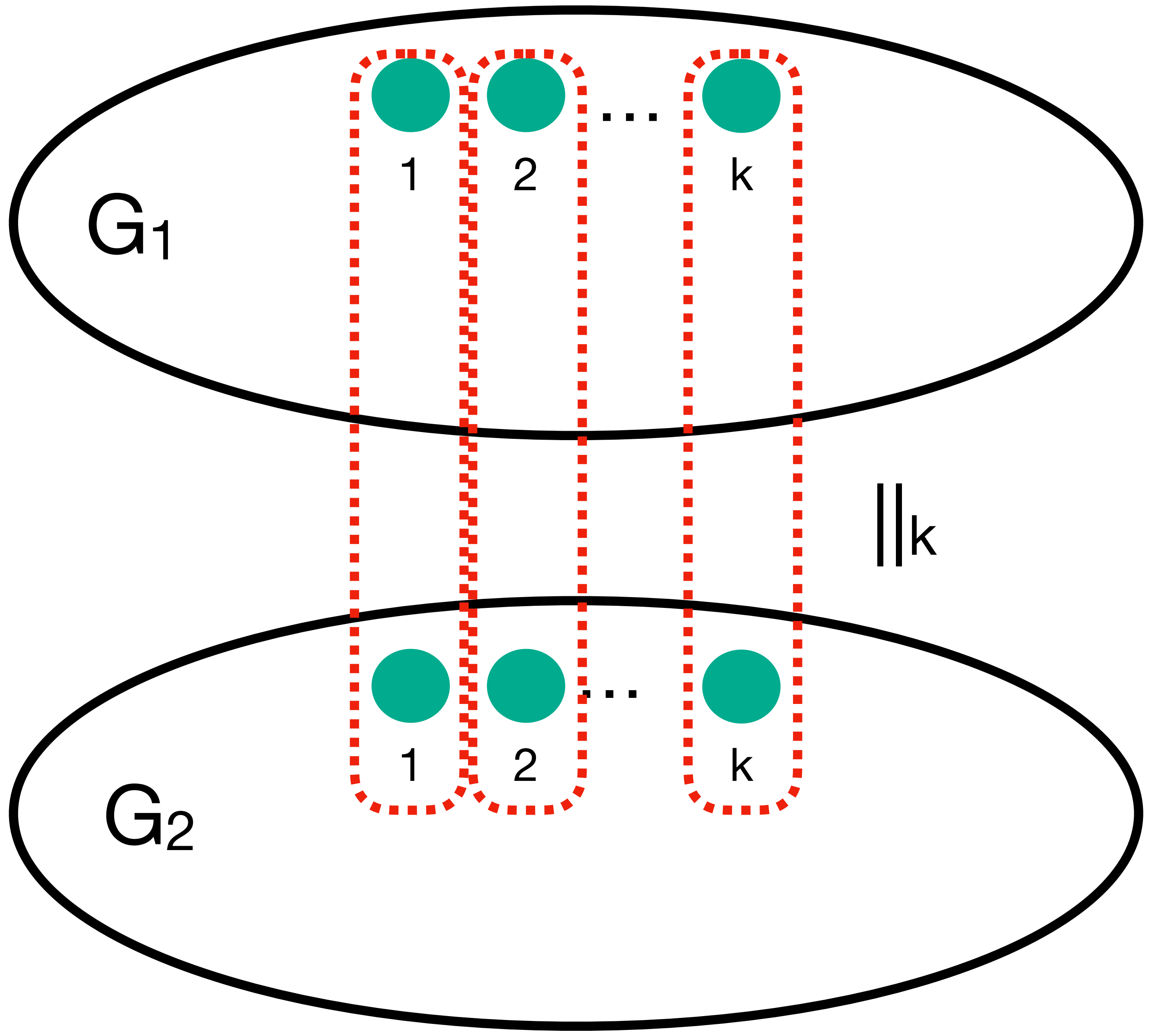

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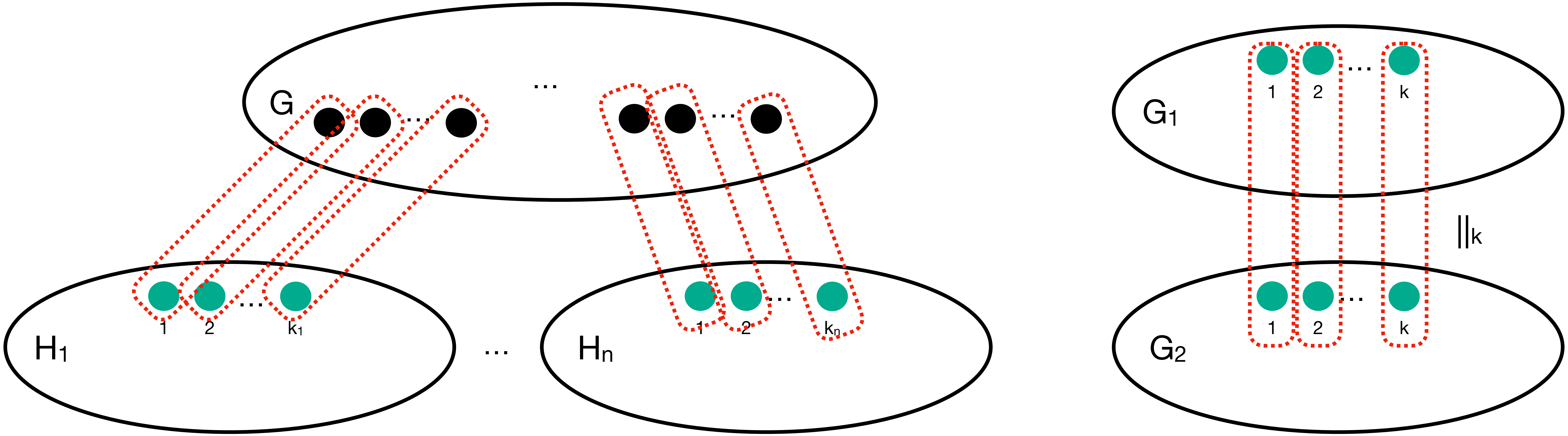


Grammar rules of the form $u \to v \|_k w$ or $u \to (G, v_1, \ldots v_n)$

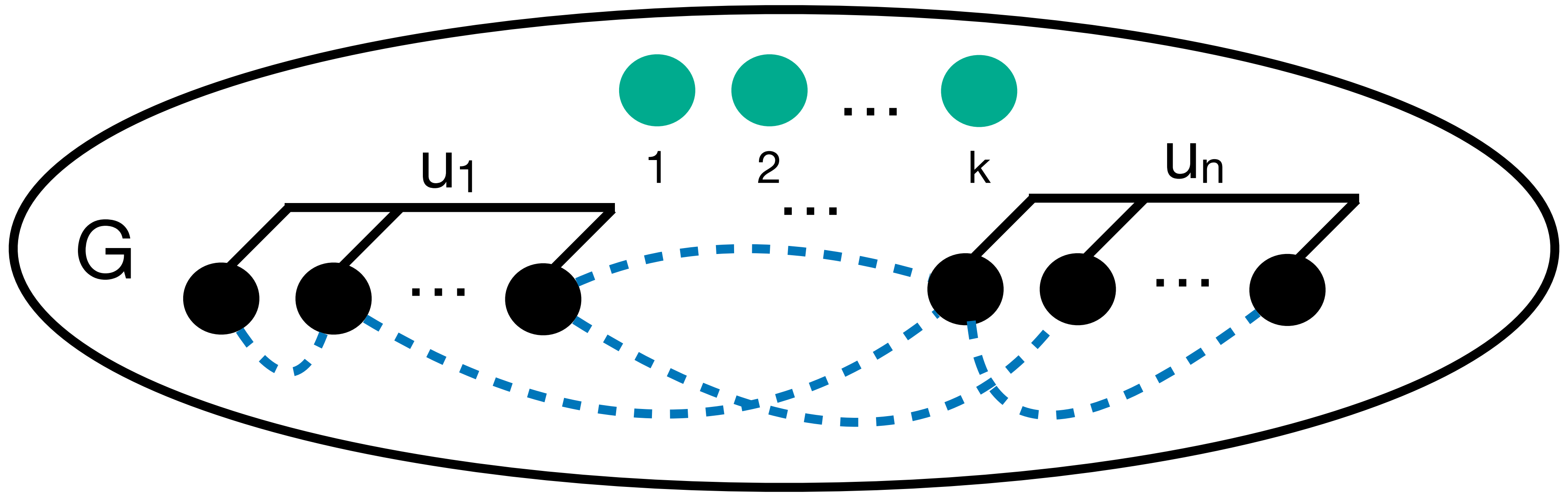
A context-free graph language is a component of the least solution (with rules viewed as set constraints)

Regular Graph Grammars

Hyperedge-replacement (HR) grammars with operations of the form $(G,u_1,\ldots,u_n)$ and $||_k$

Additional conditions on each $(G,u_1,\ldots,u_n)$ [Courcelle'91]

1. G has at least one edge
    - either a single terminal edge with only sources attached,
    - or at least one internal vertex on each edge
2. Any two vertices are linked by a terminal and internal path



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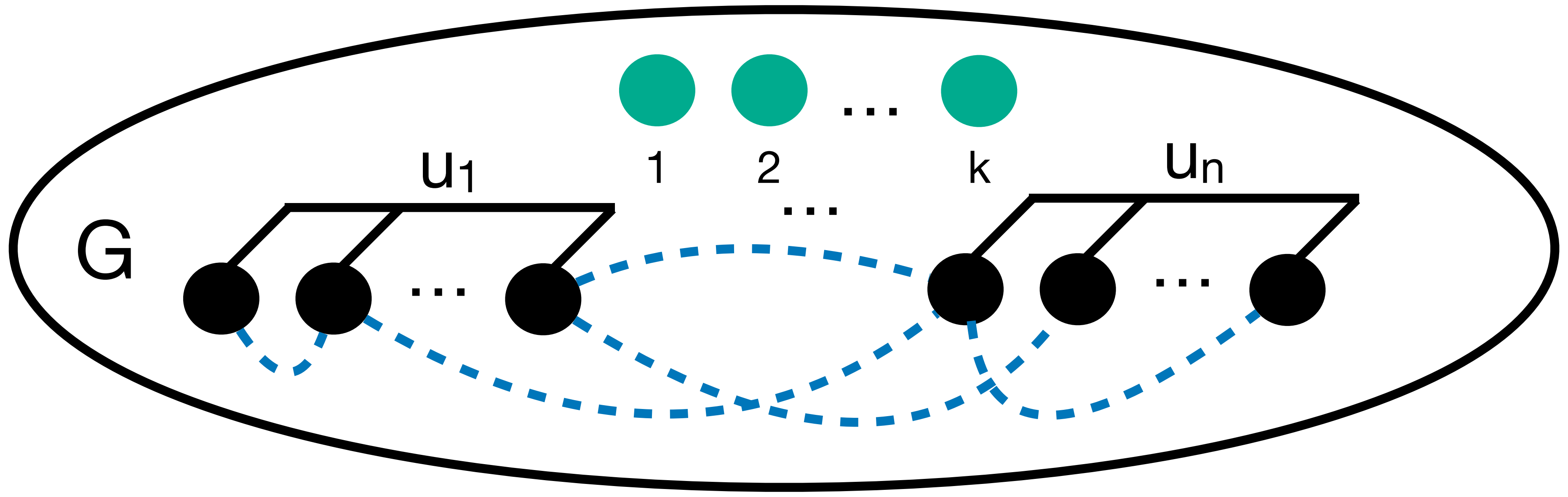

Three types of rules, where U and W are disjoint sets of nonterminals:

- $u \rightarrow u \,||_k\, w$, $u \in U$, $w \in W$
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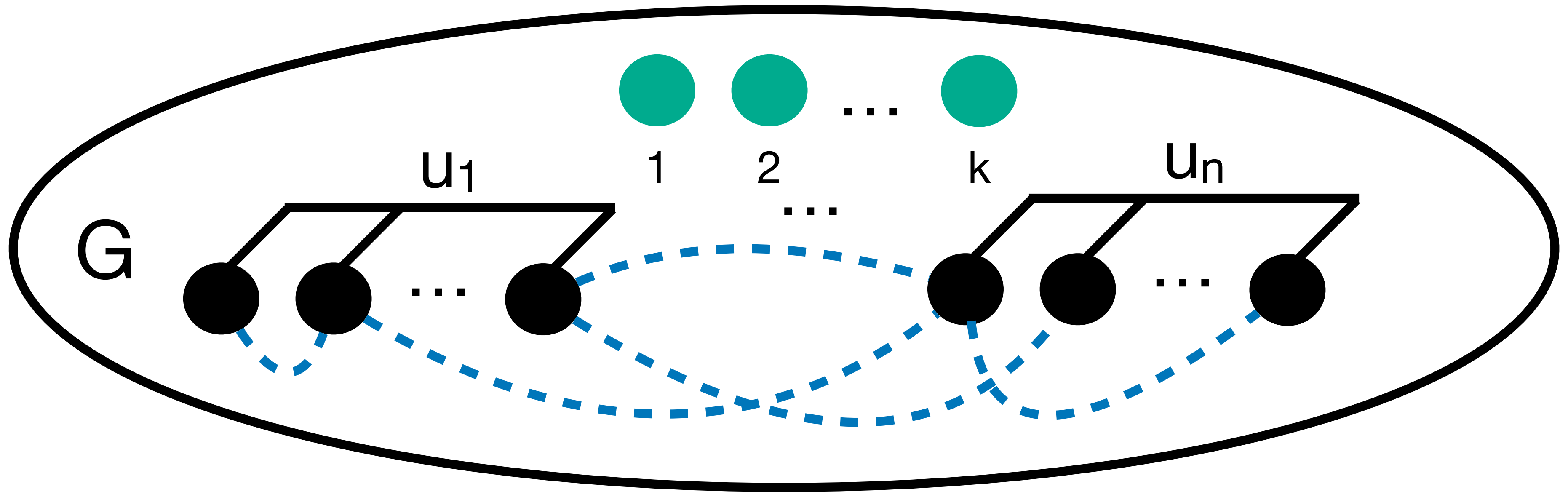

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The context-free sets produced by regular graph grammars are MSO-definable [Courcelle'92]

(Regular) Grammars vs (Regular) SIDs

$$u \rightarrow (G, v_1, \ldots v_n)$$

$$P(\underbrace{x_1, \ldots x_{\#P}}_{\text{sources}}) \leftarrow \underbrace{\exists y_1 \ldots \exists y_m}_{\text{internal vertices}} .\ \psi * \pmb{\ast}_{i=1..n}\, Q_i(\underbrace{z_{i,1}, \ldots, z_{i,\#Q_i}}_{\text{nonterminal edges}})$$

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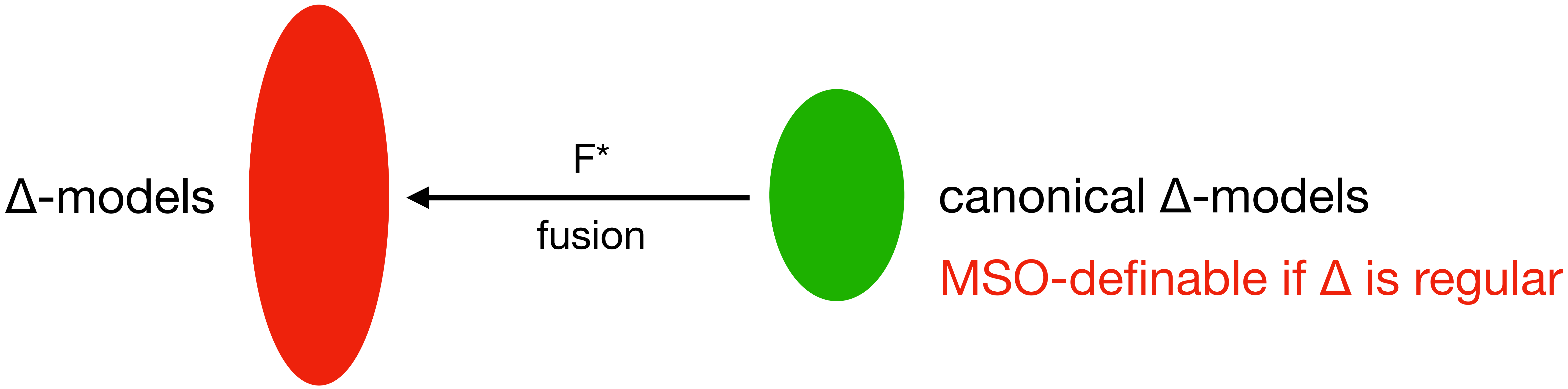


Δ-models

F*

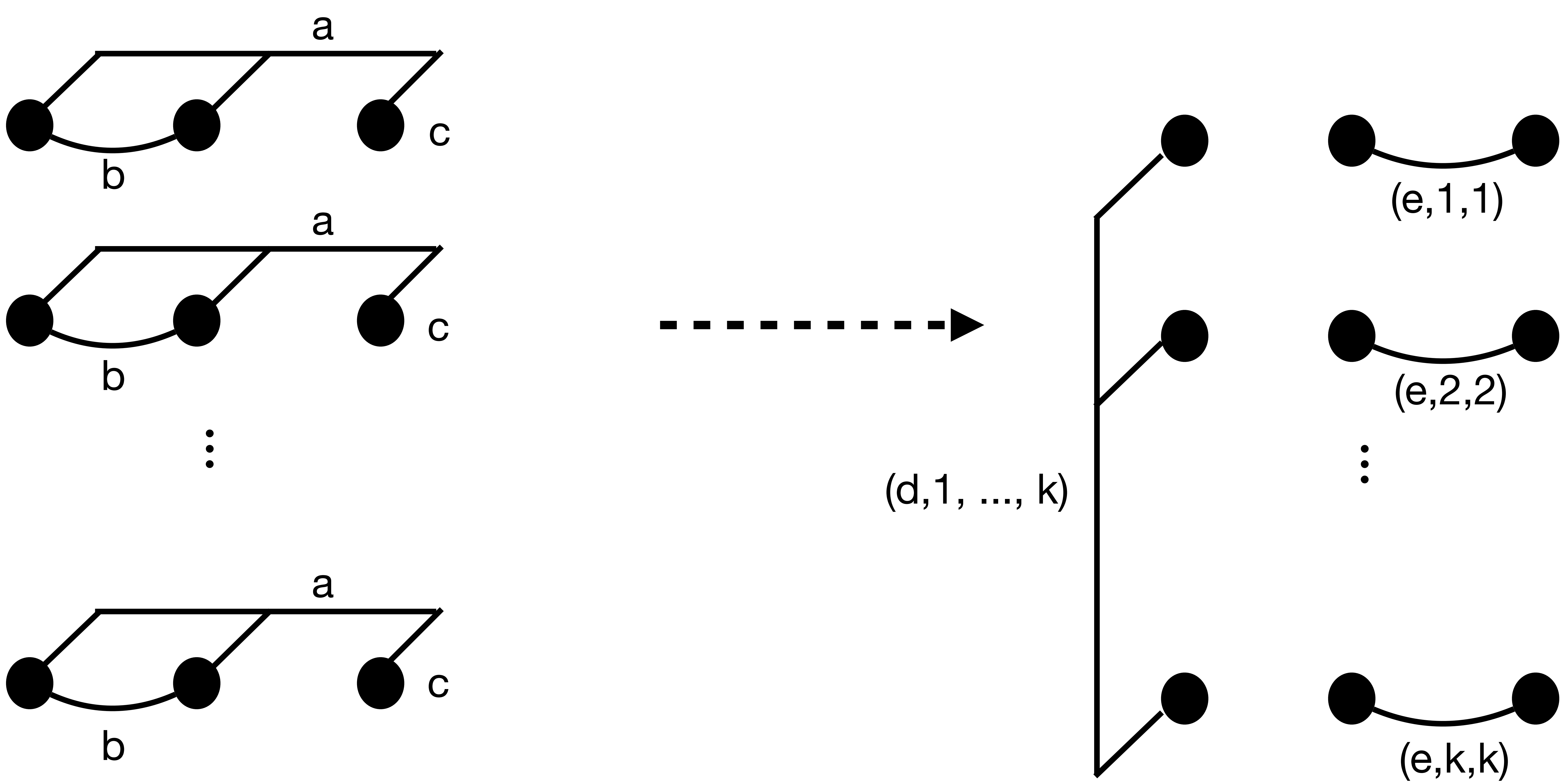
fusion

canonical Δ-models

MSO-definable if Δ is regular

Definable Transductions

k layers =
copies of the input structure



$\mathbb{R}$={a,b,c}

$\mathbb{R}'$={d,e}

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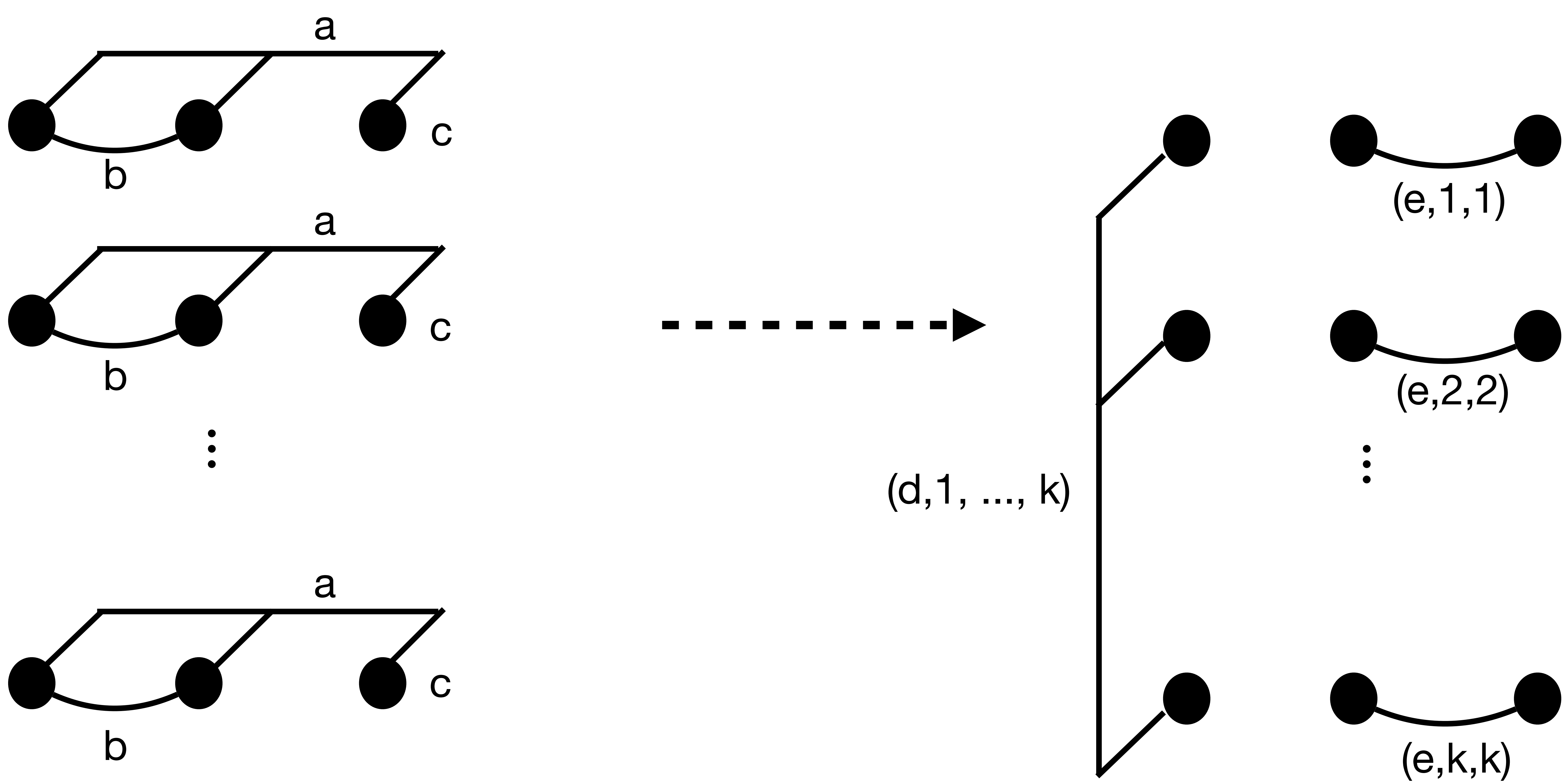

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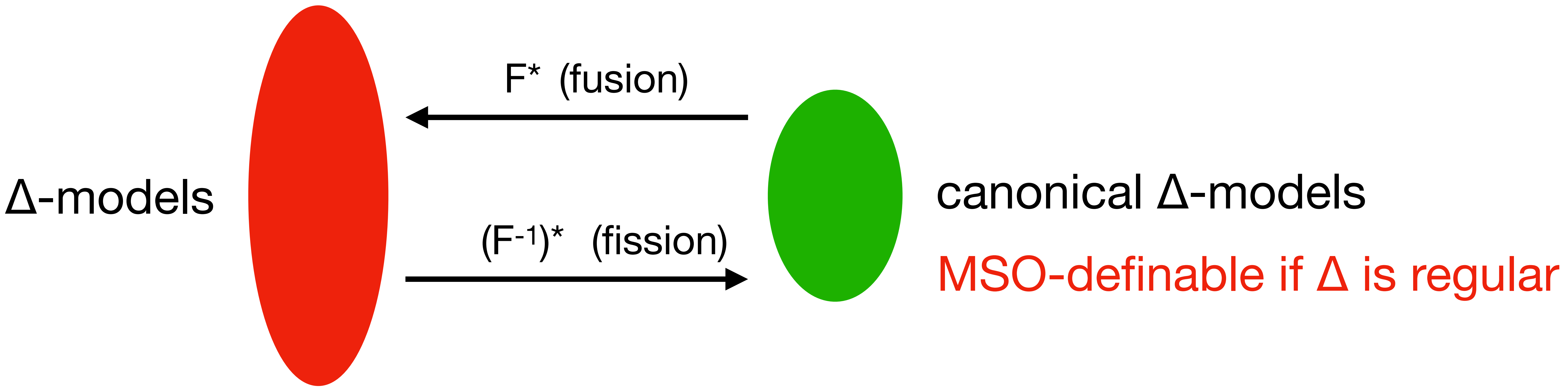
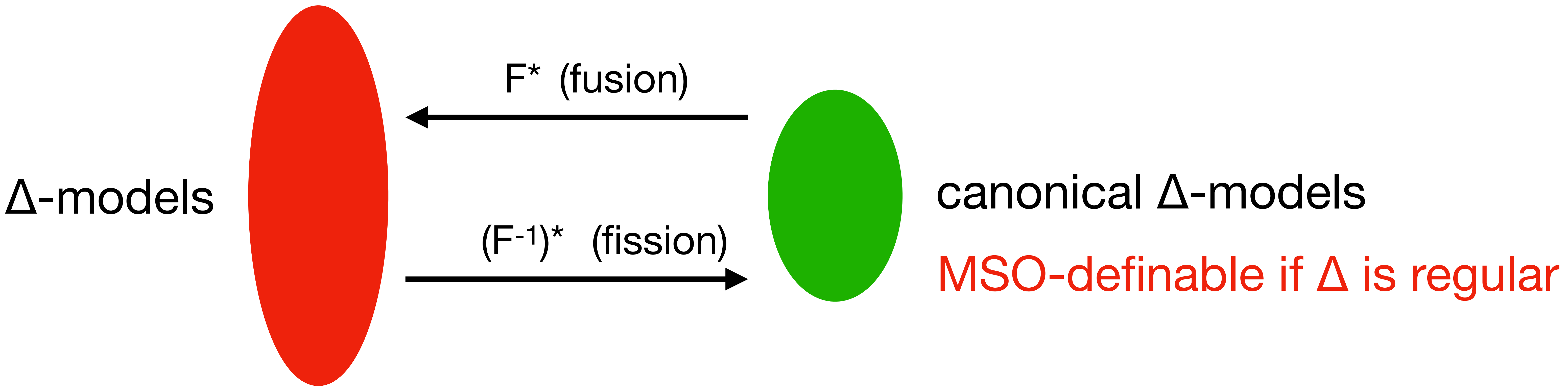
If L’ ⊆ Struc($\mathbb{R}'$) is MSO-definable and R is a definable $\mathbb{R}$-$\mathbb{R}'$ transduction then R^{-1}(L') ⊆ Struc($\mathbb{R}$) is MSO-definable

MSO-Definable Sets of Models



MSO-definable if Δ is regular

MSO-Definable Sets of Models



F^{-1} is a definable transduction, but $(F^{-1})^*$ is (provably) not, in general

- transduction scheme that uses quantification over sets of edges

For a regular SID Δ, assuming that the set of Δ-models of a given sentence has bounded tree-width, this set is obtained from the set of canonical Δ-models by applying F^k, for a bounded k≥1

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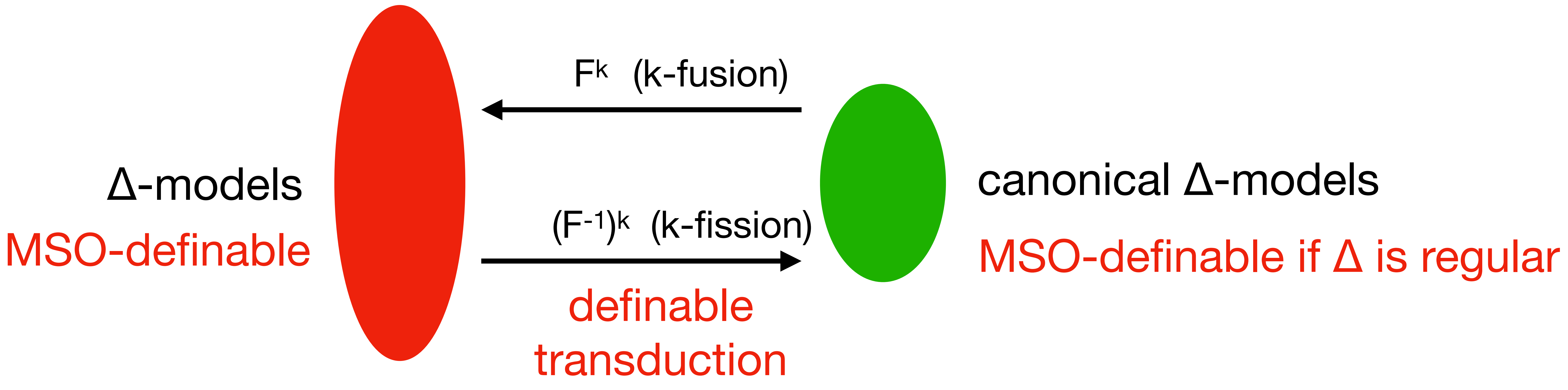

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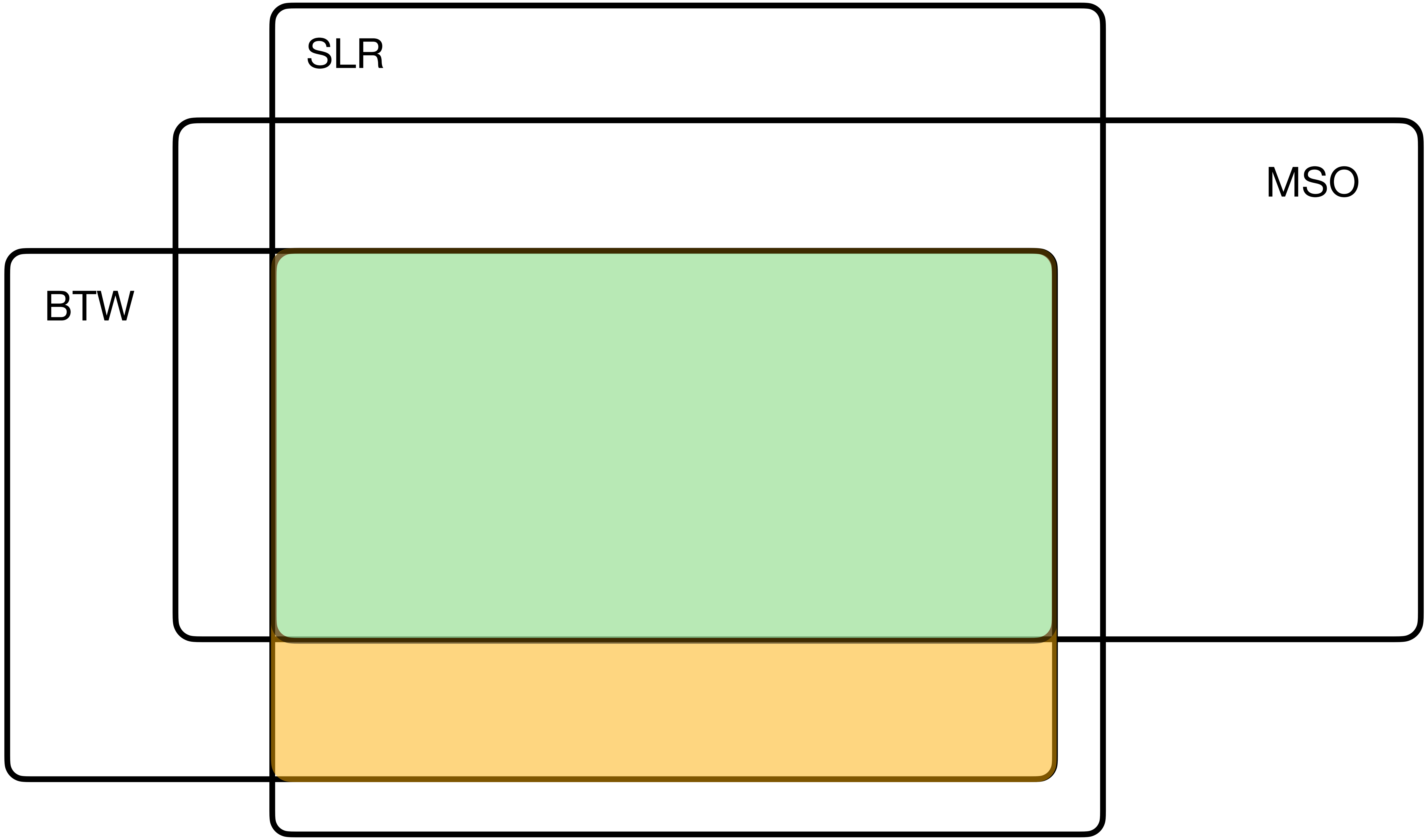

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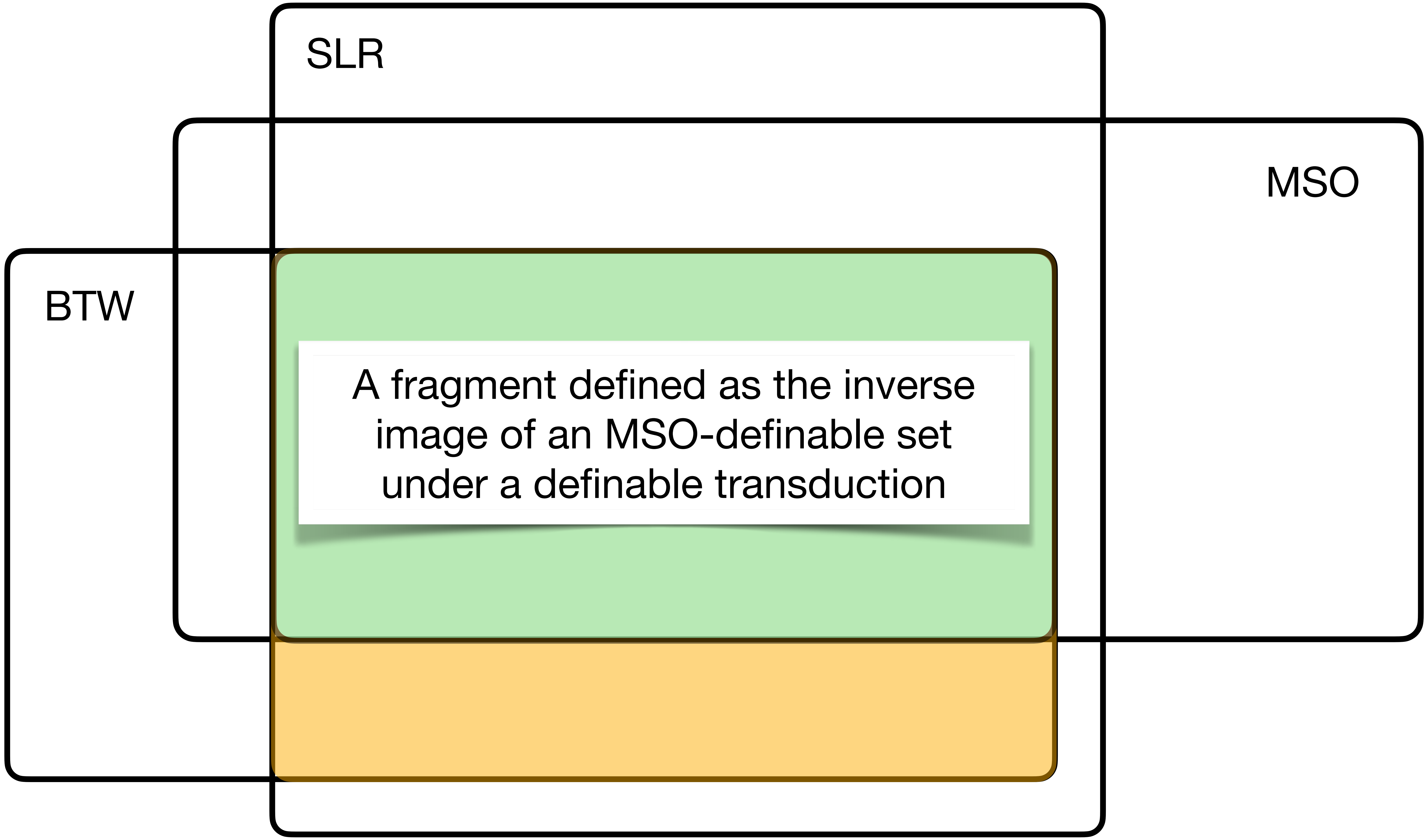

A fragment defined as the inverse image of an MSO-definable set under a definable transduction

Conclusions and Future Work

A definition of a large fragment of SLR that describes MSO-definable and tree-width bounded sets of structures

- the idea can be used starting with other MSO-definable HR grammars (e.g., series-parallel graphs)

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Future Work

- A grammar-based characterization of HR and (C)MSO-definable sets
- Complexity for entailments between SLR ∩ BTW ∩ CMSO sets